Quantum Mechanics crash course

(For the scholar with an higher education in mathematics)

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Contents

1 Mathematical definitions

General references: [CTDL05, CCP82, Sha94]

1.1 Hilbert space

We define a special (euclidean) vector space, which will be called *Hilbert space*. We omit here the complete definition and list of properties of this special vector space.

Let's just mention one aspect: the dimension. An Hilbert space can have finite dimension or infinite dimension. Which dimension should be used for the Hilbert space that describes a physical (quantum) system depends on the system, and in particular on its degrees of freedom.

1.1.1 Infinite dimension

The subatomic particles (e.g. electrons) must be described using an Hilbert space with infinite dimensions. In this case the elements of the space can be represented as complex functions:

$$\mathcal{H} : \{\psi : \mathbb{R}^3 \mapsto \mathbb{C}\} \tag{1}$$

In this case the scalar product is defined as:

$$(\psi, \phi) \equiv \int \psi^*(\vec{r}) \cdot \phi(\vec{r}) \, \mathrm{d}^3\vec{r} \tag{2}$$

A lot of efforts have to be made, to carefully define the support of this vector space properly. In particular, if we use the set of functions integrable according to Reimann, it turns out that the space would not be complete. So a new type of integral is needed (the Lebesgue integral), and we need to include in the space not only the functions but also the *distributions* (e.g. the Dirac delta 'function').

1.1.2 Finite dimension

Some more simple quantum systems are described by an Hilbert space with finite dimensions.This is in particular the case with the qubit, the quantum system at the center of the quantum information/computation theory. This system is described by an Hilbert space of dimension d=2.

The elements of this space will be represented as *column vectors*, while the elements of the *dual* of the vector space will be represented as row vectors. In this case the *scalar product* is defined as

$$(\psi, \phi) \equiv (a_1^*, a_2^*) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \equiv a_1^* b_1 + a_2^* b_2 \tag{3}$$

where:

$$\psi, \phi \in \mathcal{H}; \quad \psi = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}; \quad \phi = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{4}$$

and we have used the "row times column" matrix product.

Sometimes, the representation as columns and rows is used also for the infinite-dimensional case, when this is more intuitive. In the case of infinite dimension we use ellipses, as in

$$\psi, \phi \in \mathcal{H}, \psi = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix}, \phi = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix}. \tag{5}$$

1.2 Operators on the Hilbert space

We will need linear operators defined on the Hilbert space:

$$\hat{O} : \mathcal{H} \mapsto \mathcal{H}. \tag{6}$$

A simple (and important) example is the *position operator*. If we consider the simplified case of the Hilbert space of the one dimensional functions:

$$\mathcal{H} = \{\psi : \mathbb{R} \mapsto \mathbb{C}\} \tag{7}$$

we define

$$\hat{X} : \psi(x) \mapsto x \cdot \psi(x) \tag{8}$$

This operator is sometimes called “multiplication operator”, because it just takes the function and multiplies it for its independent variable. The 3D version is a “collection of operators”, each multiplying for a different component: $(\hat{X}, \hat{Y}, \hat{Z})$.

Another example is the “derivative operator”. Again in the simplified case of the Hilbert space of the one dimensional functions, we define

$$\hat{D} : \psi(x) \mapsto \frac{\mathrm{d}\psi(x)}{\mathrm{d}x} \tag{9}$$

1.3 Dirac notation

It is useful to introduce the following notation. We denote an element of the Hilbert space as:

$$|\psi\rangle \in \mathcal{H} \tag{10}$$

and an element of the dual space as

$$\langle\psi| \in \mathcal{H}_d \tag{11}$$

The first is called a *ket*, and the second is called a *bra*.

The scalar product will be, in this notation:

$$(\psi, \phi) \equiv \langle\psi|\phi\rangle \tag{12}$$

1.4 Eigen-basis representation

We now turn our attention to the *eigenvectors* of a given linear operator. If we consider the $\hat{X}$ operator, we will designate with $\{|x\rangle\}_{x\in\mathbb{R}}$ the collection of its eigenvectors. It is possible to show that $\hat{X}$ is an *hermitian* operator, so that:

- its collection of eigenvectors is an orthonormal basis for the Hilbert space (called *eigenbasis* of $\hat{X}$)
- all its eigenvalues are real, and in particular
- its eigenvalues are x, i.e. $\forall x \in \mathbb{R}, \hat{X}\,|x\rangle = x\,|x\rangle$

So, we can imagine that the *abstract vector* $|\psi\rangle \in \mathbb{H}$ has as components, in the representation of the eigenbasis $\{|x\rangle\}_{x\in\mathbb{R}}$ of $\hat{X}$, the complex values:

$$\{\langle x|\psi\rangle\}_{x\in\mathbb{R}} = \{\psi(x)\}_{x\in\mathbb{R}} \tag{13}$$

This means that, if we want to visualize the vector, or "ket" $|\psi\rangle$ as a column, in the representation of the eigenbasis of the position operator, we have:

$$|\psi\rangle = \begin{pmatrix} \psi(x_1) \\ \psi(x_2) \\ \psi(x_3) \\ \vdots \end{pmatrix} \tag{14}$$

where $(x_1, x_2, x_3, \dots)$ change with continuity over $\mathbb{R}$. Of course this is an "hand-waving" improper way of talking. Nevertheless is useful for the intuition.
As an exemple, this shows how the Hilbert space has to be infinite-dimensional.

Another linear operator is the impulse operator $\hat{P}_x$ defined as:

$$\hat{P}\,|\psi\rangle \equiv \frac{\partial\psi}{\partial x} \tag{15}$$

1.5 Representation of an operator

If the vectors are represented as column arrays, and the co-vectors (elements of the dual space) are represented as row arrays, the linear operators can be represented as matrices.

Of course, a representation is given *in a specified basis*. If we represent the operator $\hat{X}$ in its eigenbasis the matrix elements will be

$$\left(\hat{X}\right)_{x,x'} = \langle x|\,\hat{X}\,|x'\rangle \tag{16}$$

Of course, this matrix will be diagonal, as can be shown using the orthonormality of the eigenbasis.

This gives also the origin of the names *bra* and *ket*, since they encompass in a bracket the operator.

2 Postulates of QM

2.1 Introduction: state and observable

We need to look carefully at how we describe a system. In classical mechanics we have some parameters, and a certain (well chosen) collection of values of these parameters tells us all we need to know about the system, i.e. we have a full description of the system.

In quantum mechanics we have to separate two concepts, which in this classical framework are "mixed up".

On one hand, we want to consider "the state of the system", and on the other hand, we want to consider the possible parameters which can be measured. We will say that a system is in a certain state, which is an abstract thing and contains all the information about the system, and then we decide to "interrogate" the state with respect to a specific parameter. We call the parameter "an observable" of the system. To perform a measurement means to choose the observable, and interrogate the state of the system with respect to that observable.

2.2 The postulates

1. each state of a quantum system is described by an element of an Hilbert space

2. each observable is represented by an hermitian operator over the same Hilbert space

3. the only possible results of a measurement of that observable are represented by the eigenvalues of the operator $\hat{O}$. In particular, if the system under measurement is in the state $|\psi\rangle$, the probability that the measurement of $\hat{O}$ gives as result o is given by $|\langle o|\psi\rangle|^2$. Right after the measurement the system will be in the corresponding eigen-state $|o\rangle$.

4. the evolution of a quantum system is given by the Shrödinger equation:

$$\hat{H}|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle \tag{17}$$

   where the operator $\hat{H}$ is the one representing the energy of the system

As a first application, we give a *physical meaning* to the values ψx of a ket $|\psi\rangle$: $|\psi(x)|^2$ is the probability for the system to be found in the position x.

References

[CCP82] Piero Caldirola, Renzo Cirelli, and Giovanni M Prosperi. *Introduzione alla fisica teorica*. Utet, 1982.

[CTDL05] Claude Cohen-Tannoudji, Bernard Diu, and Frank Laloe. *Quantum Mechanics (vol.1)*, volume 1. WILEY-VCH, wiley-vch edition, 2005.

[Sha94] Ramamurti Shankar. *Principles of quantum mechanics*. Springer Science & Business Media, 1994.