Rotations

1 Rotations in general

A rotation is in general a transformation of the 3D space with the following properties:

1. does not change the distances between positions
2. does not change the origin of the frame of reference
3. does not change the orientation of the frame of reference

Such transformations of $\mathbb{R}^3$ can be represented as 3×3 matrices with *real* elements. These transformations can be combined, and it's possible to show that they form a group.
It is possible to show that a transformation R which leave distances unchanged is represented by a, *orthogonal* matrix. We remember that a matrix is orthogonal iff its transposed is equal to its inverse:

$$RR^T = \mathbb{I}. \tag{1}$$

Moreover, is possible to show that if a $\mathbb{R}^3 \mapsto \mathbb{R}^3$ transformation preserves the reference axes orientation, it representing matrix has determinant $= 1$

$$det(R) = 1 \tag{2}$$

We can conclude that the group of rotations on $\mathbb{R}^3$ is isomorphic to the group $SO(3)$, i.e. the *special* group of 3×3 orthogonal matrices (special meaning det=1)

2 Two dimentions

In two dimensions (i.e. in $\mathbb{R}^2$), if we represent positions as column vectors $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, an element of $SO(2)$ can be written as a 2×2 matrix and, because of the algebraic constrains, the elements can be written as trigonometric functions of the same parameter.

In particular, a rotation of an angle θ around the origin, in the *counter-clockwise* sense, will be written as :

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3}$$

while a rotation of the same angle in the *clockwise* sense will be:

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{4}$$

3 Three dimensions

Rotations in $3D$ do not commute. This means that if we reverse a given sequence of rotations we get in general a different outcome. This also implies that we can not compose two rotations by adding their corresponding angles. In other words, the "triplets of angles" such as the *Euler angles* or the *roll-pitch-yaw* angles, are not vectors!

3.1 Rotations around the coordinate axes

Rotations in $\mathbb{R}^3$, of an angle θ around the three axes, in *counter-clockwise* sense if observed with the axis pointing toward the observer, are:

$$R_{\hat{i}}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \tag{5}$$

$$R_{\hat{j}}(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{6}$$

$$R_{\hat{k}}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7}$$

3.2 Rotations around an arbitrary axis

In general is possible to rotate around any axis. If the axis is specified by a versor (unit vector)

$$\hat{u} = (u_x, u_y, u_z) \tag{8}$$

(with $\sqrt{u_x^2 + u_y^2 + u_z^2} = 1$) then the rotation of an angle θ around $\vec{u}$ in the counter-clockwise sense is

$$R_{\hat{u}}(\theta) = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix} \sin\theta + (\mathbb{I} - \hat{u}\hat{u}^T)\cos\theta + \hat{u}\hat{u}^T \tag{9}$$

where

$$\hat{u}\hat{u}^T = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} (u_x, u_y, u_z) = \begin{pmatrix} u_x^2 & u_x u_y & u_x u_z \\ u_y u_x & u_y^2 & u_y u_z \\ u_z u_x & u_z u_y & u_z^2 \end{pmatrix}. \tag{10}$$

Developing

$$R_{\hat{u}}(\theta) = \begin{pmatrix} u_x^2(1-\cos\theta)+\cos\theta & u_x u_y(1-\cos\theta) - u_z \sin\theta & u_x u_z(1-\cos\theta) + u_y \sin\theta \\ u_x u_y(1-\cos\theta) + u_y \sin\theta & u_y^2(1-\cos\theta)+\cos\theta & u_y u_z(1-\cos\theta) - u_x \sin\theta \\ u_x u_z(1-\cos\theta) - u_y \sin\theta & u_y u_z(1-\cos\theta) + u_x \sin\theta & u_z^2(1-\cos\theta)+\cos\theta \end{pmatrix} \tag{11}$$

3.3 generalization to n dimensions (nested dimensions)

this is taken from wikipedia, and has to be re-elaborated

A 3?3 rotation matrix like:

$$Q_{3\times3} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12}$$

suggests a 2×2 rotation matrix:

$$Q_{2\times 2} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{13}$$

is embedded in the upper left corner: :

$$Q_{3\times 3} = \begin{pmatrix} Q_{2\times 2} & \vec{0} \\ \vec{0}^{\,T} & 1 \end{pmatrix}. \tag{14}$$

This is no illusion; not just one, but many, copies of "n"-dimensional rotations are found within ("n"+1)-dimensional rotations, as [[subgroup]]s. Each embedding leaves one direction fixed, which in the case of 3×3 matrices is the rotation axis. For example, we have

$$Q_{\hat{i}}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \tag{15}$$

$$Q_{\hat{j}}(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \tag{16}$$

:

$$Q_{\hat{k}}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{17}$$

fixing the x axis, the y axis, and the z axis, respectively. The rotation axis need not be a coordinate axis; if $u = (x, y, z)$ is a unit vector in the desired direction, then :

$$Q_{\vec{u}}(\theta) = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \sin\theta + (I - \hat{u}\hat{u}^T)\cos\theta + \hat{u}\hat{u}^T \tag{18}$$

$$= \begin{pmatrix} (1-x^2)c_\theta + x^2 & -zs_\theta - xyc_\theta + xy & ys_\theta - xzc_\theta + xz \\ zs_\theta - xyc_\theta + xy & (1-y^2)c_\theta + y^2 & -xs_\theta - yzc_\theta + yz \\ -ys_\theta - xzc_\theta + xz & xs_\theta - yzc_\theta + yz & (1-z^2)c_\theta + z^2 \end{pmatrix} \tag{19}$$

$$= \begin{pmatrix} x^2(1-c_\theta) + c_\theta & xy(1-c_\theta) - zs_\theta & xz(1-c_\theta) + ys_\theta \\ xy(1-c_\theta) + zs_\theta & y^2(1-c_\theta) + c_\theta & yz(1-c_\theta) - xs_\theta \\ xz(1-c_\theta) - ys_\theta & yz(1-c_\theta) + xs_\theta & z^2(1-c_\theta) + c_\theta \end{pmatrix}, \tag{20}$$

where $c_\theta = \cos\theta$, $s_\theta = \theta$, is a rotation by angle θ leaving axis "'u"' fixed.
A direction in ("n"+1)-dimensional space will be a unit magnitude vector, which we may consider a point on a generalized sphere, S^n. Thus it is natural to describe the rotation group SO$(n+1)$ as combining SO(n) and S^n. A suitable formalism is the *fiber bundle*, SO$(n) \hookrightarrow$ SO$(n+1) \mapsto S^n$ where for every direction in the "base space", S^n, the "fiber" over it in the "total space", SO$(n+1)$, is a copy of the "fiber space", SO(n), namely the rotations that keep that direction fixed.
Thus we can build an $n \times n$ rotation matrix by starting with a 2×2 matrix, aiming its fixed axis on S^2 (the ordinary sphere in three-dimensional space), aiming the resulting rotation on S^3, and so on up through S$^{n?1}$. A point on S^n can be selected using "n" numbers, so we again have "n"("n"?1)/2 numbers to describe any $n \times n$ rotation matrix. In fact, we can view the sequential angle decomposition, discussed previously, as reversing this process. The composition of "n"?1 Givens rotations brings the first column (and row) to (1,0,. . . ,0), so that the remainder of the matrix is a rotation matrix of dimension one less, embedded so as to leave (1,0,. . . ,0) fixed.

4 Infinitesimal rotations

An important property is that infinitesimal rotations are *commutative*. [...]

5 Relationships between $SO(3)$, $SU(2)$, the rotations, the angular momenta, the Pauli matrices and the spin