Quantum Optics notes

(personal collected notes)

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Contents

Chapter 1

Introduction

In this document I collect and try to have an organic rewriting of the quantum optics theory and its formalism.

- The first section, in chapter 5 is from [...].
- The second section, in chapter 2, is from [...].
- The chapter 8 is about the "operative" beam-splitter formalism (Heisemberg representation) and is from the work on Quantum Random Number Generators done with Fréd.
- The chapter 7 is from the work done with Pieter Kok in Oxford. It extends on a rarther large appendix on the calculations
- The chapter 3 is from the undergrad thesis (discussions with Antonino Chiummo), and is made of two sections, one on the beam splitter and the quadratures, the other on the omodina. Listen also to a later recording from Frédéric Grosshans on the same subject.
- The chapter 4 is a collection of my notes, taken from [BR04] (maybe also taken from "fisica molecoare" QED introductory lectures) (last modified on 15 May 2007).

Part I

Note in italiano

Chapter 2

Quantizzazione del campo

2.1 Equazione delle onde

(vedi appunti di Ottica Quantistica, file “ottica classica”)

2.2 Seconda Quantizzazione

2.2.1 Modi di Slater

La prima cosa da fare, ancora in formalismo classico, è introdurre i modi normali della radiazione. Per fissare le idee possiamo pensare agli ‘automodi’ in una cavità, ossia alla ‘base’ di modi di oscillazione del campo elettromagnetico che si hanno in una cavità. Comunque questo è solo un caso particolare: con altre configurazioni (condizioni al contorno) si hanno altri automodi.

In generale diciamo che il campo elettrico e il campo magnetico li possiamo sviluppare su una certa base di ‘modi fondamentali’ :

$$\left\{ \begin{array}{lcl} \vec{E}\,(\vec{r},t) & = & -\frac{1}{\sqrt{\epsilon}} \sum_a p_a(t)\, \vec{E}_a\,(\vec{r}) \\ \vec{H}\,(\vec{r},t) & = & \frac{1}{\sqrt{\mu}} \sum_a \omega_a\, q_a(t)\, \vec{H}_a\,(\vec{r}) \end{array} \right. \tag{2.1}$$

(modi di Slater)

dove per ora possiamo considerare le p e le q che compaiono qui come semplici coefficienti, mentre a è un indice cumulativo di tutti gli indici necessari per individuare i modi.

Assumiamo che valgono delle relazioni di ortonormalità per questi (auto)modi :

$$\left\{ \begin{array}{lcl} \int_{vol.cav.} \vec{E}_a(\vec{r}) \cdot \vec{E}_b(\vec{r})\, dV & = & \delta_{a\,b} \\ \int_{vol.cav.} \vec{H}_a(\vec{r}) \cdot \vec{H}_b(\vec{r})\, dV & = & \delta_{a\,b} \end{array} \right. \tag{2.2}$$

(ortonormalità dei modi)

Adesso, in vista della quantizzazione, scriviamo l'energia del campo elettromagnetico all'interno della cavità (hamiltoniana) in funzione dei campi. La scriviamo come integrale della densità di energia esteso al volume della cavità :

$$\mathcal{H}_{e.m.} = \int_V \left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2 \right) dV \tag{2.3}$$

ora esplicitiamo i campi come sviluppo sui modi di Slater :

$$\mathcal{H}_{e.m.} = \int_V \left[\frac{1}{2}\epsilon \left(-\frac{1}{\sqrt{\epsilon}} \sum_a p_a(t)\, \vec{E}_a(\vec{r}) \right)^2 + \frac{1}{2}\mu \left(-\frac{1}{\sqrt{\mu}} \sum_a p_a(t)\, \vec{H}_a(\vec{r}) \right)^2 \right] dV \tag{2.4}$$

Portiamo l'integrazione sotto il segno di serie, e tiriamo fuori dall'integrale i termini non spaziali :

$$\mathcal{H}_{e.m.} = \frac{1}{2} \left[\sum_a p_a^2(t) \int_V \vec{E}_a(\vec{r})\ \vec{E}_a(\vec{r})\, dV + \sum_a \omega_a^2\, q_a^2(t) \int_V \vec{H}_a(\vec{r})\ \vec{H}_a(\vec{r})\, dV \right] \tag{2.5}$$

usando le proprietà di ortonormalità, tutti gli integrali valgono 1 :

$$\mathcal{H}_{e.m.} = \sum_a \frac{1}{2} \left[p_a^2(t) \ + \ \omega_a^2 q_a^2(t) \right] \tag{2.6}$$

(Hamiltoniana classica)

Qua succede un fatto cruciale : riconosciamo in quest'espressione una *somma di hamiltoniane di oscillatore armonico* :

$$\mathcal{H}_a(p,q) = \frac{1}{2}\left[p_a^2(t) \; + \; \omega_a^2 q_a^2(t)\right]. \tag{2.7}$$

Ricordiamo che le p e le q non sono necessariamente posizione e impulso, ma semplici coefficienti dello sviluppo sui modi normali. Tuttavia riconosciamo l'uguaglianza formale con l'Hamiltoniana di oscillatore armonico :

$$\mathcal{H}_{e.m.} = \sum_a \mathcal{H}_a(p,q). \tag{2.8}$$

Questa è dunque l'Hamiltoniana del sistema "radiazione elettromagnetica in una cavità".

2.2.2 Equazioni di Hamilton

Si dimostra che le equazioni di Hamilton per questo sistema :

$$\begin{cases} \dot{p}_a &= -\frac{\partial \mathcal{H}_{e.m.}(p,q)}{\partial q_a} = -\omega_a^2\, q_a \\ \dot{q}_a &= -\frac{\partial \mathcal{H}_{e.m.}(p,q)}{\partial p_a} = p_a \end{cases} \tag{2.9}$$

(equazioni di Hamilton)

sono equivalenti alle *equazioni di Maxwell.*

Dunque le equazioni di Maxwell si possono risolvere espandendo i campi sui modi di Slater e risolvendo le equazioni di Hamilton.

2.2.3 Promozione di p e q

A questo punto avviene la quantizzazione : detto con le parole di Dirac ≪promuoviamo le p_a e le q_a ad operatori (hermitiani)≫ :

$$p_a(t) \;\rightarrow\; \hat{p}_a(t) \tag{2.10}$$

$$q_a(t) \;\rightarrow\; \hat{q}_a(t) \tag{2.11}$$

Di questi operatori non si dà l'espressione 'esplicita' (ad esempio come operatori differenziali, o come operatori di moltiplicazione), ma si dà solo la loro algebra, cioè le regole di commutazione :

$$[\hat{q}_a(t)\,,\,\hat{p}_b(t)] \;=\; i\,\hbar\,\delta_{a\,b} \tag{2.12}$$

$$[\hat{q}_a(t)\,,\,\hat{q}_b(t)] \;=\; [\hat{p}_a(t)\,,\,\hat{p}_b(t)] = 0 \tag{2.13}$$

(tra l'altro queste regole di commutazione ci dicono che l'algebra di questi operatori è completa (qualunque cosa questo significhi... andare a vedere!)).

Ribadiamo che il fatto che abbiamo usato le lettere p e q, e il fatto che c'è una 'somiglianza' con l'hamiltoniana di oscillatore armonico è solo un fatto formale. In altre parole per il momento questi 'singoli oscillatori armonici' sono solo 'associati' al campo elettromagnetico, ma non hanno un 'esistenza fisica'.

Fatta questa promozione, anche l'Hamiltoniana 'diventa' un operatore :

$$\hat{\mathcal{H}}_{e.m.} = \sum_a \hat{\mathcal{H}}_a\left(\hat{p}_a, \hat{q}_a\right). \tag{2.14}$$

dove l'Hamiltoniano di singolo oscillatore è

$$\hat{\mathcal{H}}_a(\hat{p}_a, \hat{q}_a) = \frac{1}{2}\left[\hat{p}_a^2 + \omega_a^2 \hat{q}_a^2\right]. \tag{2.15}$$

note :

- le regole di commutazione per gli operatori $\hat{p}_a$ e $\hat{q}_a$ valgono istante per istante
- l'hamiltoniano è funzione di $\hat{p}_a$ e di $\hat{q}_a$, e dunque è un operatore funzione di operatori; ma non è definita la "derivata parziale di un operatore funzione di operatori"
- tuttavia gli operatori $\hat{p}_a$ e $\hat{q}_a$ dipendono dal tempo. Dunque possiamo utilizzare la seguente 'relazione' (è un teorema? e di chi? (forse è il teorema di Hamilton...))

che lega la derivata di un operatore dipendente dal tempo con il suo commutatore con l'Hamiltoniana :

$$\frac{d\,\hat{O}}{dt} = i\hbar \left[\hat{\mathcal{H}} \,,\, \hat{O} \right] \tag{2.16}$$

per cui, le equazioni di Hamilton diventano :

$$\begin{cases} \frac{d\,\hat{p}_a}{dt} &= i\hbar \left[\hat{\mathcal{H}} \,,\, \hat{p}_a \right] = -\omega_a^2 \hat{q}_a \\ \frac{d\,\hat{q}_a}{dt} &= \left[\hat{\mathcal{H}} \,,\, \hat{q}_a \right] = \hat{p}_a \end{cases} \tag{2.17}$$

che sono dunque le equazioni che regolano le evoluzioni degli operatori $\hat{q}_a$ e $\hat{p}_a$ e che abbiamo detto essere equivalenti alle equazioni di Maxwell.

2.2.4 Quantizzaione dei campi

Avendo quantizzato le p_a e le q_a, possiamo quantizzare i campi, usando i loro sviluppi sui modi di Slater (ricordiamo che le p_a e le q_a sono i coefficienti di questi sviluppi) :

$$\begin{cases} \hat{\vec{E}}\,(\vec{r},t) &= \frac{1}{\sqrt{\epsilon}} \sum_a \hat{p}_a(t) \vec{E}\,(\vec{r}) \\ \hat{\vec{H}}\,(\vec{r},t) &= \frac{1}{\sqrt{\mu}} \sum_a \omega_a \hat{q}_a(t) \vec{H}\,(\vec{r}) \end{cases} \tag{2.18}$$

(Notare che sono stati quantizzati solo i coefficienti, che danno la dipendenza temporale, e non i modi elettromagnetici, che danno la dipendenza spaziale).

Questi operatori sono hermitiani perché lo sono gli operatori $\hat{q}_a$ e $\hat{p}_a$

Sarebbe più corretto definire questi oggetti "campi di operatori", ma vengono invece chiamati "operatori campo".

La forma esplicita di questi operatori la possiamo conoscere solo se conosciamo la forma esplicita dei modi di Slater, che a loro volta dipendono dalle "condizioni al contorno" (caratteristiche della cavità). Ad esempio, per una cavità cubica, con in più un ipotesi di periodicità all'infinito (Born - von Karman) i modi di Slater sono le onde piane.

Ripetiamo che, almeno per gli operatori $\hat{q}_a$ e $\hat{p}_a$, si tratta di 'operatori astratti', di cui non è data cioè una rappresentazione (ad es. differenziale, o altro). Tuttavia vedremo che è possibile sviluppare tutta la teoria lavorando con gli operatori in forma astratta.

Comunque abbiamo bisogno almeno di un modo per 'calcolare' i *valori di aspettazione* di questi operatori. Tra l'altro si dimostra che il valore di aspettazione di un operatore non dipende dalla rappresentazione.

Per fare questo si introducono gli operatori di creazione e distruzione $\hat{a}_a$ e $\hat{a}_a^\dagger$. Possiamo vedere questi due operatori come delle nuove variabili della dinamica hamiltoniana, ottenute a partire da $\hat{q}_a$ e $\hat{p}_a$ con una trasformazione.

Dunque introduciamo $\hat{a}_a$ e $\hat{a}_a^\dagger$ dicendo qual'è la loro relazione con $\hat{q}_a$ e $\hat{p}_a$:

$$\begin{cases} \hat{a}_a(t) &= \frac{1}{\sqrt{2\hbar\omega_a}} \left[\hat{p}_a(t) \,+\, i\,\omega_a \hat{q}_a(t)\right] \\ \hat{a}_a^\dagger(t) &= \frac{1}{\sqrt{2\hbar\omega_a}} \left[\hat{p}_a(t) \,-\, i\,\omega_a \hat{q}_a(t)\right] \end{cases} \tag{2.19}$$

$$\begin{cases} \hat{p}_a(t) &= \frac{\sqrt{2\hbar\omega_a}}{2} \left[\hat{a}_a(t) \,+\, \hat{a}_a^\dagger(t)\right] \\ \hat{q}_a(t) &= \frac{\sqrt{2\hbar\omega_a}}{2} \left[\hat{a}_a(t) \,-\, \hat{a}_a^\dagger(t)\right] \end{cases} \tag{2.20}$$

(la seconda è da verificare)

Con una procedura 'standard', già vista a istituzioni per l'oscillatore armonico, si dimostra che questi operatori alzano e abbassano l'energia dell'oscillatore, si definisce l'operatore numero, e si scrive l'Hamiltoniana in termini dell'operatore numero, e dunque degli operatori $\hat{a}_a$ e $\hat{a}_a^\dagger$.

Inoltre, a partire dall'espressione dei campi come sviluppo sui modi di Slater (che è in termini degli operatori $\hat{p}_a$ e $\hat{q}_a$) possiamo scrivere gli operatori campo in termini degli operatori $\hat{a}_a$ e $\hat{a}_a^\dagger$.

Se come modi di Slater consideriamo il caso particolare delle onde piane, l'indice a diventa l'indice $\vec{k}$ delle onde piane (vettore d'onda), al quale dobbiamo aggiungere un indice λ che trasporta l'informazione sulla polarizzazione :

$$\hat{\vec{E}}\,(\vec{r},t) = \sum_{\vec{k},\lambda} \mathcal{E}_{\vec{k},\lambda}\, \vec{\varepsilon}_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda}\, e^{-i\left(\omega_{\vec{k}}\, t + \vec{k}\cdot\vec{r}\right)} \,+\, h.c. \tag{2.21}$$

$$\hat{\vec{H}}\,(\vec{r},t) = \sum_{\vec{k},\lambda} \mathcal{E}_{\vec{k},\lambda}\,\frac{\left(\vec{k}\times\vec{\varepsilon}_{\vec{k},\lambda}\right)}{\omega_{\vec{k}}\,\mu}\,\hat{a}_{\vec{k},\lambda}\,e^{-i\left(\omega_{\vec{k}}\,t+\vec{k}\cdot\vec{r}\right)} \;+\; h.c. \tag{2.22}$$

dove:

→ $\vec{\varepsilon}_{\vec{k},\lambda}$ è un *versore* (ho messo il segno di vettore per non confonderlo con il 'cappuccio' degli operatori) che indica la *polarizzazione*

→ $\mathcal{E} \equiv \left(\frac{\hbar\,\nu_{\vec{k}}}{\varepsilon_0\,V}\right)^{1/2}$ è una costante (che ha le dimensioni di un campo elettrico)

→ **h.c.** sta per "hermitiano coniugato".

(notiamo che, anche se $\hat{a}_{\vec{k},\lambda}$ e $\hat{a}^{\dagger}_{\vec{k},\lambda}$ sono operatori non hermitiani, i campi sono operatori hermitiani)

A questo punto possiamo scrivere anche l'hamiltoniano del sistema, cioè (l'operatore che rappresenta) l'energia (elettromagnetica) del sistema non più in termini degli operatori $\hat{p}_a$ e $\hat{q}_a$ ma in termini di $\hat{a}_{\vec{k},\lambda}$ e $\hat{a}^{\dagger}_{\vec{k},\lambda}$. A partire dall'Hamiltoniana classica, scritta in termini dei campi :

$$\mathcal{H}_{e.m.} = \int_V \left(\frac{1}{2}\varepsilon\,E^2 + \frac{1}{2}\mu H^2\right) dV \tag{2.23}$$

analogamente a quanto visto prima, possiamo quantizzare esplicitando i campi, questa volta nella forma appena vista in termini di $\hat{a}_{\vec{k},\lambda}$ e $\hat{a}^{\dagger}_{\vec{k},\lambda}$:

$$\hat{\mathcal{H}} \;=\; \frac{1}{2}\int_V \left[\varepsilon\left(\sum_{\vec{k},\lambda} \mathcal{E}_{\vec{k},\lambda}\,\vec{\varepsilon}_{\vec{k},\lambda}\,\hat{a}_{\vec{k},\lambda}\,e^{-i\left(\omega_{\vec{k}}\,t+\vec{k}\cdot\vec{r}\right)} \;+\; h.c.\right)^2 + \right. \tag{2.24}$$

$$\left. +\mu\left(\sum_{\vec{k},\lambda} \mathcal{E}_{\vec{k},\lambda}\,\frac{\left(\vec{k}\times\vec{\varepsilon}_{\vec{k},\lambda}\right)}{\omega_{\vec{k}}\,\mu}\,\hat{a}_{\vec{k},\lambda}\,e^{-i\left(\omega_{\vec{k}}\,t+\vec{k}\cdot\vec{r}\right)} \;+\; h.c.\right)^2\right] dV \tag{2.25}$$

Ora, portando gli integrali dentro le serie, e sfruttando l'ortonormalità dei modi di Slater, si arriva alla forma :

$$\hat{\mathcal{H}} = \sum_{\vec{k},\lambda} \hbar\, \omega_{\vec{k}} \left(\hat{a}^{\dagger}_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda} + \frac{1}{2} \right) \tag{2.26}$$

(hamiltoniano in termini degli operatori di creazione e distruzione)

che è la somma di hamiltoniane di oscillatore armonico, e che porta al concetto di fotone.

Osserviamo che questa forma la potevamo ottenere anche applicando il 'cambio di coordinate' da $\hat{p}_a$ e $\hat{q}_a$ a $\hat{a}_{\vec{k},\lambda}$ e $\hat{a}^{\dagger}_{\vec{k},\lambda}$ all'Hamiltoniana ottenuta in prima istanza. Tuttavia mi piaceva mostrare come gli $\hat{a}_{\vec{k},\lambda}$ e $\hat{a}^{\dagger}_{\vec{k},\lambda}$ vengono dai campi.

2.2.5 Intensità

(vedi appunti scritti a mano, file "quantizzazione del campo")

Il formalismo quantistico comporta il noto "problema di ordinamento". Riassumiamolo in poche parole. Consideriamo una certa grandezza fisica che è espressa come prodotto di altre grandezze. In fromalismo classico le grandezze sono funzioni numeriche, e dunque il prodotto è commutativo. Quando si *quantizza*, si esprime la stessa grandezza per mezzo di operatori. In generale il prodotto di operatori non è commutativo. Dunque ci si trova davanti alla seguente situazione: alla stessa grandezza classica corrispondono diversi possibili modi di rappresentarla in formalismo quantistico, in quanto in questo formalismo i diversi ordini in cui si scrive l'operatore prodotto di operatori portano in generale ad operatori diversi.

Detto in un altro modo, diverse osservabili, descritte in formalismo classico con operatori, hanno come *limite classico* la stessa grandezza classica.

Un esempio di questa situazione è l'intensità del campo elettrico.

Scriviamo in formalismo classico l'intensità in termini della densità di energia elettromagnetica. L'energia elettromagnetica è:

$$W = \frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2 \tag{2.27}$$

da cui, l'intensità del campo elettromagnetico è $W \cdot \frac{c}{n}$

$$I = \frac{1}{2}\frac{c}{n}\varepsilon E^2 + \frac{1}{2}\frac{c}{n}\mu H^2. \tag{2.28}$$

Per passare al formalismo quantistico andiamo incontro a delle ambiguità. Infatti possiamo scrivere in diversi modi i quadrati di operatori. Ricordando infatti le definizioni degli operatori campo (2.21) e (2.22):

$$\hat{\vec{E}}\left(\vec{r},t\right) = \sum_{\vec{k},\lambda} \mathcal{E}_{\vec{k},\lambda}\, \vec{\varepsilon}_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda}\, e^{-i\left(\omega_{\vec{k}}\, t + \vec{k}\cdot\vec{r}\right)} \,+\, h.c.$$

$$\hat{\vec{H}}\left(\vec{r},t\right) = \sum_{\vec{k},\lambda} \mathcal{E}_{\vec{k},\lambda}\, \frac{\left(\vec{k}\times\vec{\varepsilon}_{\vec{k},\lambda}\right)}{\omega_{\vec{k}}\, \mu}\, \hat{a}_{\vec{k},\lambda}\, e^{-i\left(\omega_{\vec{k}}\, t + \vec{k}\cdot\vec{r}\right)} \,+\, h.c.$$

dove compaiono gli hermitiani coniugati.

Definiamo

$$\hat{\vec{E}}^{+}\left(\vec{r},t\right) \equiv \sum_{\vec{k},\lambda} \mathcal{E}_{\vec{k},\lambda}\, \vec{\varepsilon}_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda}\, e^{-i\left(\omega_{\vec{k}}\, t + \vec{k}\cdot\vec{r}\right)} \tag{2.29}$$

$$\hat{\vec{E}}^{-}\left(\vec{r},t\right) \equiv \sum_{\vec{k},\lambda} \mathcal{E}_{\vec{k},\lambda}\, \vec{\varepsilon}_{\vec{k},\lambda}\, \hat{a}^{\dagger}_{\vec{k},\lambda}\, e^{i\left(\omega_{\vec{k}}\, t + \vec{k}\cdot\vec{r}\right)} \tag{2.30}$$

osserviamo che questi due operatori, così come $\hat{a}^{\dagger}$ e $\hat{a}$, sono l'uno l'aggiunto dell'altro.

Dunque scriviamo il campo elettrico come

$$\hat{\vec{E}}\left(\vec{r},t\right) = \hat{\vec{E}}^{+}\left(\vec{r},t\right) \,+\, \hat{\vec{E}}^{-}\left(\vec{r},t\right). \tag{2.31}$$

Ciò posto, l'intensità del campo elettrico, che è proporzionale al quadrato del campo, la possiamo scrivere in tre modi diversi:

$$\hat{I} = \hat{\vec{E}}^{-}(\vec{r},t) \cdot \hat{\vec{E}}^{+}(\vec{r},t) \tag{2.32}$$

$$\hat{I}' = \hat{\vec{E}}^{+}(\vec{r},t) \cdot \hat{\vec{E}}^{-}(\vec{r},t) \tag{2.33}$$

$$\hat{I}'' = \hat{\vec{E}}^{+}(\vec{r},t) + \hat{\vec{E}}^{-}(\vec{r},t) \cdot \hat{\vec{E}}^{+}(\vec{r},t) + \hat{\vec{E}}^{-}(\vec{r},t)\,. \tag{2.34}$$

(qui c'è qualcosa che mi sfugge, credo riguardi le proprietà degli operatori aggiunti)

2.3 Stati Coerenti

È possibile introdurre gli stati coerenti come quegli stati del campo elettromagnetico che sono "generati" da una corrente classica (pensiamo alla corrente che scorre in un'antenna) (vedi lezione 20, t=1h 09'). Qui invece li introduciamo in maniera "formale".

Definiamo il seguente *Operatore di Spostamento*

$$\hat{D}(\alpha) \equiv e^{\alpha\,\hat{a}^{\dagger} - \alpha^{*}\,\hat{a}}. \tag{2.35}$$

Lo stato che risulta dall'applicazione di questo opreatore sullo stato di vuoto è detto *stato coerente*:

$$\hat{D}(\alpha)|0\rangle = e^{\alpha\,\hat{a}^{\dagger} - \alpha^{*}\,\hat{a}}|0\rangle = |\alpha\rangle. \tag{2.36}$$

In queste formule α è un *parametro* complesso, il cui significato fisico sarà presto chiaro.

Le proprietà degli stati coerenti sono legate alle proprietà dell'operatore di spostamento.

Definiamo i seguenti due operatori "ausiliari":

$$\begin{cases} \hat{A}(\alpha) \equiv \alpha\,\hat{a}^{\dagger} \\ \hat{B}(\alpha) \equiv -\alpha^{*}\,\hat{a} \end{cases} \tag{2.37}$$

e quindi possiamo scrivere

$$\hat{D}(\alpha) \equiv e^{\hat{A}+\hat{B}}. \tag{2.38}$$

Dalla proprietà di commutazione

$$\left[\hat{a}\,,\,\hat{a}^\dagger\right] = \mathbb{I} \tag{2.39}$$

discende

$$\begin{aligned}\left[\hat{A}\,,\,\hat{B}\right] &= \alpha\hat{a}^\dagger\left(-\alpha^*\hat{a}\right) - \alpha^*\hat{a}\left(-\alpha\hat{a}^\dagger\right) = -\alpha\alpha^*\hat{a}^\dagger\hat{a} + \alpha^*\alpha\hat{a}\hat{a}^\dagger = \\ &= |\alpha|^2\left[\hat{a}\,,\,\hat{a}^\dagger\right] = |\alpha|^2 \in \mathbb{R}.\end{aligned} \tag{2.40}$$

Introduciamo adesso il seguente teorema

Theorem 2.3.1 (Baker-Hausdorff) *Dati due generici operatori $\hat{A}$ e $\hat{B}$, se essi commutano con il loro commutatore, cioè se $\left[\hat{A}\,,\,\left[\hat{A}\,,\,\hat{B}\right]\right] = \left[\hat{B}\,,\,\left[\hat{A}\,,\,\hat{B}\right]\right] = 0$ allora si ha la seguente* relazione di Baker-Hausdorff*:*

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}\,e^{\hat{B}}\,e^{-\frac{1}{2}\left[\hat{A}\,,\,\hat{B}\right]}. \tag{2.41}$$

[. . .]

2.4 Stati miscela

(vedi appunti scritti a mano, file "Stati miscela e rappresentazioni")

Sia $\{|\psi\rangle\}$ un insieme di stati (del campo e.m.), e sia $P(\psi)$ la probabilità che il campo sia nello stato $|\psi\rangle$; sia infine $\hat{O}$ una certa osservabile. Se il sistema (il campo) si trova con certezza nello stato $|\psi\rangle$, il valor medio di $\hat{O}$ (cioè l'esito più probabile di una misura di O) è:

$$\langle\hat{O}\rangle_\psi = \langle\psi|\hat{O}|\psi\rangle. \tag{2.42}$$

Ma conoscere esattamente lo stato del campo è una condizione non realistica: a causa delle fluttuazioni (approfondire questo punto) quello che capita è che conosciamo l'insieme dei possibili stati del campo $\{|\psi\rangle\}$, e la distribuzione di probabilità $P(\psi)$. Quando la nostra conoscenza dello stato di un sistema quantistico è di questo tipo, si dice che il sistema è in uno *stato miscela*. In tal caso, se si effettuano tante misure di $\hat{O}$, il valore medio degli esiti delle misure sarà:

$$\langle \hat{O} \rangle_{\{|\psi\rangle\}\,;\,P(\psi)} = \sum_{\{|\psi\rangle\}} \langle \psi | \hat{O} | \psi \rangle \, P(\psi). \tag{2.43}$$

Chapter 3

Beam Splitter e quadrature

3.1 Beam splitter e quadrature

- **Rappresentazione della radiazione elettromagnetica**

a) Formalismo classico

Cominciamo con la descrizione della radiazione elettromagnetica.

Innanzitutto se si risolve l'equazione delle onde si ottiene la seguente forma per le soluzioni:

$$\vec{E}(\vec{r},t) = E(\vec{r},t)\,[e^{i(\vec{k}z+\omega t)} + e^{-i(\vec{k}z+\omega t)}]\,\hat{p}(\vec{r},t) \tag{3.1}$$

dove $E(\vec{r},t)$ è l'ampiezza complessa del campo, $\vec{k}$ (vettore d'onda) indica la direzione di propagazione (lungo z), e $\hat{p}$ è il versore di ploarizzazione. L'ampiezza complessa ha un modulo $E_0(\vec{r},t) \in \Re$, che descrive l'*ampiezza dell'onda*, e una fase $\phi(\vec{r},t)$, che descrive la *geometria del fronte d'onda*, nonché la *differenza di fase* (dell'andamento oscillante) rispetto ad una certa onda di riferimento :

$$E(\vec{r},t) \;=\; E_0(\vec{r},t)\,e^{i\,\phi(\vec{r},t)}. \tag{3.2}$$

Vogliamo ora passare ad un'altra notazione che tenga separati questi due significati della fase.

Riscriviamo la (3.1) mettendo una differenza di fase esplicita ϕ_0 all'argomento della parte oscillante:

$$\vec{E}(\vec{r},t) = E_0(\vec{r},t)\, e^{i\,\phi(\vec{r},t)}\,[e^{i(\vec{k}z+\omega t+\phi_0)} + e^{-i(\vec{k}z+\omega t+\phi_0)}] \tag{3.3}$$

raggruppando differentemente

$$\vec{E}(\vec{r},t) = E_0(\vec{r},t)\,[e^{i\,\phi(\vec{r},t)}e^{i(\vec{k}z+\omega t+\phi_0)} + e^{i\,\phi(\vec{r},t)}e^{-i(\vec{k}z+\omega t+\phi_0)}]$$

$$\vec{E}(\vec{r},t) = E_0(\vec{r},t)\,[e^{i\,[\vec{k}z+\phi(\vec{r},t)]}e^{i(\omega t+\phi_0)} + e^{i\,[\vec{k}z+\phi(\vec{r},t)]}e^{-i(\omega t+\phi_0)}] \tag{3.4}$$

passando alla notazione trigonometrica (Eulero):

$$\vec{E}(\vec{r},t) = E_0(\vec{r},t)\,[e^{i\,[\vec{k}z+\phi(\vec{r},t)]}e^{i(\omega t+\phi_0)} + e^{i\,[\vec{k}z+\phi(\vec{r},t)]}e^{-i(\omega t+\phi_0)}] \tag{3.5}$$

[...] qua ancora non ho capito i passaggi... [...]

$$\vec{E}(\vec{r},t) = E(\vec{r},t)\ \sin[\vec{k}z+\phi(\vec{r},t)]\ \sin[\omega t+\phi_0] \tag{3.6}$$

e, applicando la formula di somma trigonometrica $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$$\begin{aligned} \vec{E}(\vec{r},t) &= E_0(\vec{r},t)\,[\sin\phi_0\cos\omega t + \cos\phi_0\sin\omega t]\ \ \sin[\vec{k}z+\phi(\vec{r},t)] \\ \vec{E}(\vec{r},t) &= E_0(\vec{r},t)\,[X_1\cos\omega t + X_2\sin\omega t]\ \ \sin[\vec{k}z+\phi(\vec{r},t)] \end{aligned} \tag{3.7}$$

dove abbiamo posto

$$\frac{X_1}{X_2} \equiv \tan\phi_0. \tag{3.8}$$

Dunque in questo modo abbiamo un'altra notazione per ampiezza e fase, in termini delle quantità X_i, dette *quadrature*. Queste le possiamo immaginare come le componenti rettangolari di un piano a due dimensioni in cui viene rappresentata l'ampiezza complessa (piano dei fasori o *piano di Gauss*). Dunque descrivere il campo oscillante in termini di ampiezza e fase (ampiezza complessa, ma con la sola fase ϕ_0) significa usare le coordinate polari di questo piano, mentre passare alle quadrature significa passare alle coordinate rettangolari di questo piano. Possiamo sia immaginare l'ampiezza complessa che ruota in questo piano con velocità angolare costante (vedi pulsazione), oppure possiamo pensare alla "parte costante dell'ampiezza complessa".

Osserviamo che se si pone $X_2 = 0$ l'onda è un coseno. Possiamo prendere questa onda cosinusoidale come riferimento per la fase, e questo equivale a prendere come riferimento per gli angoli l'asse X_1 nel piano di Gauss X_1, X_2.

b) Formalismo quantistico

Richiamiamo brevemente la quantizzazione del campo elettromagnetico. Si comincia con lo scrivere l'energia elettromagnetica (per semplicità riferiamoci ad una radiazione contenuta in una cavità).

[...] seconda quantizzazione [...]

- campo in una cavità (parallelepipedica) - modi di Slater - hamiltoniana - si riconosce la forma dell'hamiltoniana come somma di hamiltoniane di oscillatori armonici - si descrive il sistema come un insieme di oscillatori armonici

[...]

Dunque in questo formalismo uno stato del sistema "campo elettromagnetico" si può rappresentare tramite un elemento dello spazio di Fock, (*stati di Fock* o *stati numero*). Tali stati si possono definire come gli autostati dell'operatore numero, ovvero, come prodotto di un (infinito) numero di tali autostati, in quanto in linea di principio i livelli energetici degli oscillatori armonici sono in numero infinito (sebbene in generale solo un numero finito ha numero di occupazione diverso da zero). Questa rappresentazione è in qualche modo antiintuitiva: abbiamo detto di descrivere il campo elettromagnetico come un insieme di oscillatori armonici, uno per ogni modo elettromagnetico (vedi modi di Slater). A sua volta ognuno di questi oscillatori può essere in uno degli infiniti livelli energetici discreti (quantizzazione). La rappresentazione degli stati numero, anziché dire che il il k-esimo oscillatore (k-esimo modo) è nel livello eccitato l, dice che nel livello l-esimo si trovano n-oscillatori (n-modi).

Questa rappresentazione degli stati del campo, ha il vantaggio di derivare direttamente dalla teoria, cioè dal modo in cui si "quantizza" il campo. Tuttavia è difficile rappresentare in questo formalismo uno stato con fase fissata.

(per queste cose sugli stati coerenti ho preso dai miei appunti di ottica quantistica) Una rappresentazione degli stati del campo EM alternativa alla rappresentazione con gli stati di Fock, è la rappresentazione con gli stati coerenti. Il fatto di passare da una rappresentazione ad un'altra può essere visto come un cambio di riferimento dello spazio di Fock (controllare...). L'insieme degli stati coerenti è un insieme completo, ma non è ortogonale.

Gli stati coerenti si possono definire come autostati dell'operatore di annichilazione $a^\dagger$:

$$a^\dagger|\alpha\rangle = \alpha|\alpha\rangle \tag{3.9}$$

α, oltre ad essere l'autovalore corrispondente, è un parametro (complesso, $\alpha \in \mathbb{C}; \alpha = |\alpha|e^{\phi}$) che caratterizza lo stato coerente.

Un modo alternativo di introdurre gli stati coerenti è quello di definirli cme gli stati che si ottengono applicando l'operatore di spostamento

$$\hat{D}(\alpha) \equiv e^{-\alpha^* \hat{a} + \alpha \hat{a}^\dagger}. \tag{3.10}$$

Infine, si può dimostrare che gli stati coerenti sono gli stati del campo EM in presenza di una corrente classica (antenna).

Se si misura il vettore campo elettrico di uno stato coerente $|\alpha\rangle$, il valore di aspettazione è una funzione oscillante (sinusoidale o cosinusoidale) del tempo, la cui ampiezza è data dal modulo di α e la cui fase è data dalla fase di α

- **Il beam splitter**

Seguendo le diverse possibili descrizioni della radiazione, anche la fisica del beam splitter la si può descrivere in formalismo classico (ottica ondulatoria), in formalismo quantistico con la rappresentazione degli stati di Fock (fotoni, rappresentazione corpuscolare) e in formalismo quantistico con la rappresentazione degli stati coerenti (rappresentazione ondulatoria).

3.2 Omodina

3.3 Notes on omodina detection

Notes taken from Fred, on 11 gen 2011 (cfr audio recording)

$$E_{in} = X \cos \omega t + Y \sin \omega t \tag{3.11}$$

$$E_{LO} = E_{LO}^0 \cos \omega t \tag{3.12}$$

$$E_{LO} \pm E_{in}$$
$$E_{LO}^2 \pm 2E_{LO}E_{in} + E_{in}^2$$
$$4E_{LO}E_{in}$$

Omodina output is the product of the local oscillator times the component of the "input signal" which has the same frequency of the local oscillator. In other words the omodina selectively amplifies inly the component of the input signal which has the same frequency of the LO.

Chapter 4

Quantizzazione del campo (II)

4.1 Modi elettromagnetici

4.1.1 Modi di Slater

La prima cosa da fare, ancora in formalismo classico, è introdurre i modi normali della radiazione. Per fissare le idee possiamo pensare agli 'automodi' in una cavità, ossia alla 'base' di modi di oscillazione del campo elettromagnetico che si hanno in una cavità. Comunque questo è solo un caso particolare : con altre configurazioni si hanno altri automodi.

In generale diciamo che il campo elettrico e il campo magnetico li possiamo sviluppare su una certa base di 'modi fondamentali' :

$$\begin{cases} \vec{E}(\vec{r},t) = -\frac{1}{\sqrt{\varepsilon}} \sum_a p_a(t)\, \vec{E}_a(\vec{r}) \\ \vec{H}(\vec{r},t) = -\frac{1}{\sqrt{\mu}} \sum_a \omega_a q_a(t)\, \vec{H}_a(\vec{r}) \end{cases} \tag{4.1}$$

(modi di Slater)

dove per ora possiamo considerare le p e le q che compaiono qui come semplici coefficienti, mentre a è un indice cumulativo di tutti gli indici necessari per individuare i modi.

Assumiamo che valgono delle relazioni di ortonormalità per questi (auto)modi :

$$\begin{cases} \int_{vol.cav.} \vec{E}_a(\vec{r}) \cdot \vec{E}_b(\vec{r}) dV = \delta_{ab} \\ \int_{vol.cav.} \vec{H}_a(\vec{r}) \cdot \vec{H}_b(\vec{r}) dV = \delta_{ab} \end{cases} \tag{4.2}$$

(ortonormalità dei modi)

4.1.2 Energia del campo

Adesso, in vista della quantizzazione, scriviamo l'energia del campo elettromagnetico all'interno della cavità (hamiltoniana) in funzione dei campi. La scriviamo come integrale della densità di energia esteso al volume della cavità :

$$\mathcal{H}_{e.m.} = \int_V \left(\frac{1}{2} \varepsilon \, E^2 + \frac{1}{2} \mu \, H^2 \right) dV \tag{4.3}$$

ora esplicitiamo i campi come sviluppo sui modi di Slater :

$$\begin{aligned} \mathcal{H}_{e.m.} &= \int_V \left[\tfrac{1}{2} \varepsilon \left(-\tfrac{1}{\sqrt{\varepsilon}} \sum_a p_a(t) \, \vec{E}_a(\vec{r}) \right)^2 + \right. \\ &\left. + \tfrac{1}{2} \mu \left(-\tfrac{1}{\sqrt{\mu}} \sum_a \omega_a q_a(t) \, \vec{H}_a(\vec{r}) \right)^2 \right] dV \end{aligned} \tag{4.4}$$

Portiamo l'integrazione sotto il segno di serie, e tiriamo fuori dall'integrale i termini non spaziali :

$$\begin{aligned} \mathcal{H}_{e.m.} &= \tfrac{1}{2} \left[\sum_a p_a^2(t) \int_V \vec{E}_a(\vec{r}) \vec{E}_a(\vec{r}) \, dV + \right. \\ &\left. + \sum_a \omega_a^2 q_a^2(t) \int_V \vec{H}_a(\vec{r}) \vec{H}_a(\vec{r}) \, dV \right] \end{aligned} \tag{4.5}$$

usando le proprietà di ortonormalità, tutti gli integrali valgono 1 :

$$\mathcal{H}_{e.m.} = \sum_a \frac{1}{2} \left[p_a^2(t) + \omega_a^2 \, q_a^2(t) \right] \tag{4.6}$$

(Hamiltoniana classica)

Qua avviene il passaggio cruciale: riconosciamo in quest'espressione una somma di hamiltoniane di oscillatore armonico:

$$\mathcal{H}_a \equiv \frac{1}{2}\left[p_a^2(t) + \omega_a^2\, q_a^2(t)\right]. \tag{4.7}$$

Ricordiamo che le p e le q non sono necessariamente posizione e impulso, ma semplici coefficienti dello sviluppo sui modi normali. Tuttavia riconosciamo l'uguaglianza formale con l'Hamiltoniana di oscillatore armonico:

$$\mathcal{H}_{e.m.} = \sum_a \mathcal{H}_a\,(p\,,\,q)\,. \tag{4.8}$$

Questa è dunque l'Hamiltoniana del sistema "radiazione elettromagnetica in una cavità".

Si dimostra che le *equazioni di Hamilton* per questo sistema :

$$\begin{cases} \dot{p}_a = -\frac{\partial\,\mathcal{H}_{em}(q,p)}{\partial\,q_a} = -\omega_a^2\,q_a \\ \dot{q}_a = -\frac{\partial\,\mathcal{H}_{em}(q,p)}{\partial\,p_a} = p_a \end{cases} \tag{4.9}$$

(equazioni di Hamilton)

sono equivalenti alle *equazioni di Maxwell.*

Dunque le equazioni di Maxwell si possono risolvere espandendo i campi sui modi di Slater e risolvendo le equazioni di Hamilton.

4.1.3 ≪Promozione≫

A questo punto avviene la quantizzazione : detto con le parole di Dirac ≪promuoviamo le p_a e le q_a ad operatori (hermitiani)≫ :

$$\begin{aligned} p_a(t) &\to \hat{p}_a(t) \\ q_a(t) &\to \hat{q}_a(t). \end{aligned} \tag{4.10}$$

Di questi operatori non si dà l'espressione 'esplicita' (ad esempio come operatori differenziali, o come operatori di moltiplicazione), ma si dà solo la loro algebra, cioè le regole di commutazione :

$$\begin{aligned} &[\hat{q}_a(t), \hat{p}_b(t)] = i\hbar\delta_{ab} \\ &[\hat{q}_a(t), \hat{q}_b(t)] = [\hat{p}_a(t), \hat{p}_b(t)] = 0 \end{aligned} \tag{4.11}$$

(tra l'altro queste regole di commutazione ci dicono che l'algebra di questi operatori è completa).

Ribadiamo che il fatto che abbiamo usato le lettere p e q, e il fatto che c'è una 'somiglianza' con l'hamiltoniana di oscillatore armonico è solo un fatto formale. In altre parole per il momento questi 'singoli oscillatori armonici' sono solo 'associati' al campo elettromagnetico, ma non hanno un' 'esistenza fisica'.

Fatta questa promozione, anche l'Hamiltoniana 'diventa' un operatore :

$$\hat{\mathcal{H}}_{em} = \sum_a \hat{\mathcal{H}}_a \left(\hat{q}_a \, , \, \hat{p}_a\right) \tag{4.12}$$

dove l'Hamiltoniano di singolo oscillatore è

$$\hat{\mathcal{H}}_a \left(\hat{q}_a \, , \, \hat{p}_a\right) \equiv \frac{1}{2} \left[\hat{p}_a^2 + \omega_a^2 \, \hat{q}_a^2\right] \tag{4.13}$$

note :

- le regole di commutazione per gli operatori $\hat{p}_a$ e $\hat{q}_a$ valgono istante per istante

- l'hamiltoniano è funzione delle p_a e delle q_a, e dunque è un operatore funzione di operatori; tuttavia non è definita la "derivata parziale di un operatore funzione di operatori"

- tuttavia gli operatori $\hat{p}_a$ e $\hat{q}_a$ dipendono dal tempo. Dunque possiamo utilizzare la seguente relazione che lega la derivata di un operatore dipendente dal tempo con il suo commutatore con l'Hamiltoniana :

$$\frac{d\,\hat{O}}{d\,t} = i\,\hbar \left[\hat{\mathcal{H}} \, , \, \hat{O}\right] \tag{4.14}$$

per cui, le equazioni¡ di Hamilton diventano :

$$\begin{cases} \frac{d\hat{p}_a}{dt} = i\hbar\left[\hat{\mathcal{H}}\,,\,\hat{p}_a\right] = -\omega_a^2\,\hat{q}_a \\ \frac{d\hat{q}_a}{dt} = \left[\hat{\mathcal{H}}\,,\,\hat{q}_a\right] = \hat{p}_a \end{cases} \tag{4.15}$$

che sono dunque le equazioni che regolano le evoluzioni degli operatori $\hat{p}_a$ e $\hat{q}_a$, e che abbiamo detto essere equivalenti alle equazioni di Maxwell.

4.2 Quantizzazione dei campi

Avendo quantizzato le p_a e le q_a, possiamo quantizzare i campi, usando i loro sviluppi sui modi di Slater (ricordiamo che le p_a e le q_a sono i coefficienti di questi sviluppi) :

$$\begin{cases} \hat{\vec{E}}(\vec{r},t) = -\frac{1}{\sqrt{\varepsilon}}\sum_a \hat{p}_a(t)\,\vec{E}_a(\vec{r}) \\ \hat{\vec{H}}(\vec{r},t) = -\frac{1}{\sqrt{\mu}}\sum_a \omega_a\,\hat{q}_a(t)\,\vec{H}_a(\vec{r}) \end{cases} \tag{4.16}$$

(notare che sono stati quantizzati solo i coefficienti, che danno la dipendenza temporale, e non i modi elettromagnetici, che danno la dipendenza spaziale)

Questi operatori sono hermitiani perché lo sono gli operatori $\hat{p}_a$ e $\hat{q}_a$.

Sarebbe più corretto definire questi oggetti “campi di operatori”, ma vengono invece chiamati “operatori campo”.

La forma esplicita di questi operatori la possiamo conoscere solo se conosciamo la forma esplicita dei modi di Slater, che a loro volta dipendono dalle “condizioni al contorno” (caratteristiche della cavità). Ad esempio, per una cavità cubica, con in più un ipotesi di periodicità all'infinito (Born - von Karman) i modi di Slater sono le onde piane.

Ripetiamo che, almeno per gli operatori $\hat{p}_a$ e $\hat{q}_a$, si tratta di ‘operatori astratti’, di cui non è data cioè una rappresentazione (ad es. differenziale, o altro). Tuttavia vedremo che è possibile sviluppare tutta la teoria lavorando con gli operatori in forma astratta.

Però abbiamo bisogno almeno di un modo per ‘calcolare’ i *valori di aspettazione* di questi operatori. Tra l'altro si dimostra che il valore di aspettazione di un operatore non dipende dalla rappresentazione.

Per fare questo si introducono gli operatori di creazione e distruzione $\hat{a}_a$ e $\hat{a}_a^\dagger$. Possiamo vedere questi due operatori come delle nuove variabili della dinamica hamiltoniana, ottenute a partire da $\hat{p}_a$ e $\hat{q}_a$ con una trasformazione.

Dunque introduciamo $\hat{a}_a$ e $\hat{a}_a^\dagger$ dicendo qual'è la loro relazione con $\hat{p}_a$ e $\hat{q}_a$:

$$\begin{cases} \hat{a}_a(t) = \frac{1}{\sqrt{2\hbar\omega_a}} \left[\hat{p}_a(t) + i\,\omega_a \hat{q}_a(t)\right] \\ \hat{a}_a^\dagger(t) = \frac{1}{\sqrt{2\hbar\omega_a}} \left[\hat{p}_a(t) - i\,\omega_a \hat{q}_a(t)\right] \end{cases} \tag{4.17}$$

$$\begin{cases} \hat{p}_a(t) = \frac{\sqrt{2\hbar\omega_a}}{2} \left[\hat{a}_a(t) + i\,\hat{a}_a^\dagger(t)\right] \\ \hat{q}_a(t) = \frac{\sqrt{2\hbar\omega_a}}{2\,i\,\omega_a} \left[\hat{a}_a(t) - i\,\hat{a}_a^\dagger(t)\right] \end{cases} \tag{4.18}$$

Con una procedura 'standard', che si usa per la trattazione dell'oscillatore armonico, si dimostra che questi operatori alzano e abbassano l'energia dell'oscillatore, si definisce l'operatore numero, e si scrive l'Hamiltoniana in termini dell'operatore numero, e dunque degli operatori $\hat{a}_a$ e $\hat{a}_a^\dagger$.

Inoltre, a partire dall'espressione dei campi come sviluppo sui modi di Slater (che è in termini degli operatori $\hat{p}_a$ e $\hat{q}_a$) possiamo scrivere gli operatori campo in termini degli operatori $\hat{a}_a$ e $\hat{a}_a^\dagger$.

Se come modi di Slater consideriamo il caso particolare delle onde piane, l'indice 'a' diventa l'indice $\vec{k}$ delle onde piane (vettore d'onda), al quale dobbiamo aggiungere un indice λ che esprime l'informazione sulla polarizzazione :

$$\begin{cases} \hat{\vec{E}}(\vec{r},t) = \sum_{\vec{k},\lambda} \mathcal{E}_{\vec{k},\lambda}\, \vec{\varepsilon}_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda}\, e^{-i\left(\omega_{\vec{k}} + \vec{k}\cdot\vec{r}\right)} + h.c. \\ \hat{\vec{H}}(\vec{r},t) = \sum_{\vec{k},\lambda} \mathcal{E}_{\vec{k},\lambda} \frac{\left(\vec{k}\times\vec{\varepsilon}_{\vec{k},\lambda}\right)}{\omega_{\vec{k}}\,\mu}\, \hat{a}_{\vec{k},\lambda}\, e^{-i\left(\omega_{\vec{k}} + \vec{k}\cdot\vec{r}\right)} + h.c. \end{cases} \tag{4.19}$$

(operatori dei campi scritti in termini di operatori di creazione e distruzione)

dove $\vec{\varepsilon}_{\vec{k},\lambda}$ è un versore che indica la polarizzazione e $\mathcal{E}_{\vec{k},\lambda} \equiv \left(\frac{\hbar\,\nu_k}{\varepsilon_0\,V}\right)^{1/2}$ è una costante (che ha le dimensioni di un campo elettrico). Si noti che, anche se $\hat{a}_{\vec{k}\lambda}$ e $\hat{a}_{\vec{k}\lambda}^\dagger$ sono operatori non hermitiani, i campi sono operatori hermitiani.

A questo punto possiamo scrivere anche l'hamiltoniano del sistema, cioè (l'operatore che rappresenta) l'energia (elettromagnetica) del sistema non più in termini degli operatori $\hat{q}_a$ e $\hat{p}_a^\dagger$ ma in termini di $\hat{a}_{\vec{k}\lambda}$ e $\hat{a}_{\vec{k}\lambda}^\dagger$. A partire dall'Hamiltoniana classica, scritta in termini dei campi :

$$\mathcal{H}_{em} = \int_V \left(\frac{1}{2}\,\varepsilon\, E^2 + \frac{1}{2}\,\mu\, H^2 \right) dV \tag{4.20}$$

analogamente a quanto visto prima, possiamo quantizzare questa *grandezza osservabile classica* esplicitando i campi, questa volta nella forma appena vista in termini di $\hat{a}_{\vec{k}\lambda}$ e $\hat{a}_{\vec{k}\lambda}^\dagger$:

$$\begin{aligned} \hat{\mathcal{H}} \;=\; & \tfrac{1}{2} \textstyle\int_V \Big[\varepsilon \Big(\sum_{\vec{k},\lambda} \mathcal{E}_{\vec{k},\lambda}\, \vec{\varepsilon}_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda}\, e^{-i\left(\omega_{\vec{k}} + \vec{k}\cdot\vec{r}\right)} + h.c. \Big)^2 + \\ & + \mu \Big(\textstyle\sum_{\vec{k},\lambda} \mathcal{E}_{\vec{k},\lambda} \frac{\left(\vec{k}\times\vec{\varepsilon}_{\vec{k},\lambda}\right)}{\omega_{\vec{k}}\,\mu}\, \hat{a}_{\vec{k},\lambda}\, e^{-i\left(\omega_{\vec{k}} + \vec{k}\cdot\vec{r}\right)} + h.c. \Big)^2 + \Big] \end{aligned} \tag{4.21}$$

e quindi, portando gli integrali dentro le serie, e sfruttando l'ortonormalità dei modi di Slater, arriviamo alla forma :

$$\hat{\mathcal{H}} = \sum_{\vec{k},\lambda} \hbar\, \omega_{\vec{k}} \left(\hat{a}_{\vec{k},\lambda}^\dagger\, \hat{a}_{\vec{k},\lambda} + \frac{1}{2} \right) \tag{4.22}$$

(hamiltoniano in termini degli operatori di creazione e distruzione)

che è la somma di hamiltoniane di oscillatore armonico, e che porta al concetto di fotone.

Osserviamo che questa forma la potevamo ottenere anche applicando un 'cambio di coordinate' da $\hat{q}_a$ e $\hat{p}_a$ a $\hat{a}_{\vec{k},\lambda}^\dagger$ e $\hat{a}_{\vec{k},\lambda}$ all'Hamiltoniana ottenuta in prima istanza. Tuttavia questa procedura mostra come gli $\hat{a}_{\vec{k},\lambda}^\dagger$ e $\hat{a}_{\vec{k},\lambda}$ siano legati ai campi.

4.3 Stati coerenti

È possibile introdurre gli stati coerenti come quegli stati del campo elettromagnetico che sono "generati" da una corrente classica, come quella che scorre in un'antenna. Tuttavia qui scegliamo di introdurre gli stati coerenti in maniera 'formale'

4.3.1 Operatore di spostamento

Introduciamo il segnete operatore, detto operatore di spostamento :

$$\hat{D} \equiv e^{\alpha\,\hat{a}^\dagger - \alpha^*\,\hat{a}} \tag{4.23}$$

('D' sta per 'displacement') dove α è un parametro complesso

Definiamo stati coerenti quegli stati ottenuti applicando questo operatore di spostamento allo stato di vuoto del campo :

$$\hat{D}(\alpha)\,|0\rangle = e^{\alpha\,\hat{a}^\dagger - \alpha^*\,\hat{a}}\,|0\rangle \equiv |\alpha\rangle. \tag{4.24}$$

Osserviamo che sia l'operatore di spostamento che lo stato coerente sono individuati dal parametro α.

Le proprietà degli stati coerenti sono legate alle proprietà dell'operatore di spostamento, che quindi andiamo ad indagare.

Definiamo i due operatori

$$\begin{cases} \hat{A} & \equiv & \alpha\,\hat{a}^\dagger \\ \hat{B} & \equiv & -\alpha^*\,\hat{a} \end{cases} \tag{4.25}$$

da cui

$$\hat{D}(\alpha)\,|0\rangle = e^{\hat{A}+\hat{B}}\,|0\rangle \equiv |\alpha\rangle. \tag{4.26}$$

Dalle proprietà di commutazione degli operatori di creazione e distruzione :

$$\left[\hat{a}\,,\,\hat{a}^\dagger\right] = 1 \tag{4.27}$$

discende

$$\begin{aligned} \left[\hat{A}\,,\,\hat{B}\right] &= \alpha\,\hat{a}^\dagger\,(-\alpha^*\,\hat{a}) - \alpha^*\,\hat{a}\left(-\alpha\,\hat{a}^\dagger\right) \\ &= -\alpha\,\alpha^*\,\hat{a}^\dagger\,\hat{a} + \alpha^*\,\alpha\,\hat{a}\hat{a}^\dagger \\ &= |\alpha|^2\left[\hat{a}\,,\,\hat{a}^\dagger\right] \end{aligned} \tag{4.28}$$

in definitiva

$$\left[\hat{A}\,,\,\hat{B}\right] \,=\, |\alpha|^2 \tag{4.29}$$

Teorema (Baker - Hausdorff)
Dati due operatori $\hat{A}$ e $\hat{B}$ che commutano con il loro commutatore :

$$\left[\hat{A}\,,\,\left[\hat{A}\,,\,\hat{B}\right]\right] \,=\, \left[\hat{B}\,,\,\left[\hat{A}\,,\,\hat{B}\right]\right] \,=\, 0 \tag{4.30}$$

si ha in tali ipotesi :

$$e^{\hat{A}+\hat{B}} \,=\, e^{\hat{A}}\,e^{\hat{B}}\,e^{-\frac{1}{2}\left[\hat{A},\hat{B}\right]} \tag{4.31}$$

Applicando questo teorema all'operatore di spostamento si ha :

$$\hat{D}(\alpha) \,=\, e^{\alpha\,\hat{a}^\dagger}\,e^{-\alpha^*\hat{a}}\;e^{-\frac{1}{2}\,|\alpha|^2} \tag{4.32}$$

(ordine normale)

$$\hat{D}(\alpha) \,=\, e^{-\alpha^*\hat{a}}\,e^{\alpha\,\hat{a}^\dagger}\;e^{\frac{1}{2}\,|\alpha|^2} \tag{4.33}$$

(ordine antinormale)

Proprietà
si dimostra che l'operatore di spostamento è un operatore unitario :

$$\hat{D}(\alpha)\,\hat{D}^\dagger(\alpha) \,=\, \hat{D}^\dagger(\alpha)\,\hat{D}(\alpha) \,=\mathbb{I} \tag{4.34}$$

inoltre si ha

$$\hat{D}^\dagger(\alpha) \,=\, \hat{D}^{-1}(\alpha) \,=\, \hat{D}(-\alpha). \tag{4.35}$$

Si dimostra che :

$$\hat{D}^{-1}(\alpha)\,\hat{a}\,\hat{D}(\alpha) \;=\; \hat{a} \,+\, \alpha. \tag{4.36}$$

Questo ci permette di dimostrare un importante proprietà degli stati coerenti, e cioè che essi sono autostati dell'operatore di distruzione $\hat{a}$. Infatti :

$$\begin{aligned} \hat{D}^{-1}(\alpha)\,\hat{a}\,\hat{D}(\alpha)\,|0\rangle \;&=\; (\hat{a} \,+\, \alpha)\;|0\rangle \\ &=\; \hat{a}\,|0\rangle \,+\, \alpha\,|0\rangle \end{aligned} \tag{4.37}$$

ma l'operatore di distruzione applicato allo stato di vuoto fa zero e dunque

$$\hat{D}^{-1}(\alpha)\,\hat{a}\,\hat{D}(\alpha)\,|0\rangle \;=\; \alpha\,|0\rangle. \tag{4.38}$$

da cui, applicando l'operatore di spostamento ad ambo i membri :

$$\begin{aligned} &\hat{D}(\alpha)\,\hat{D}^{-1}(\alpha)\,\hat{a}\,\hat{D}(\alpha)\,|0\rangle \;=\; \alpha\,\hat{D}(\alpha)\,|0\rangle \\ &\hat{a}\,\left[\hat{D}(\alpha)\,|0\rangle\right] \;=\; \alpha\,\left[\hat{D}(\alpha)\,|0\rangle\right] \\ &\hat{a}|\alpha\rangle \;=\; \alpha\,|\alpha\rangle \end{aligned}$$

abbiamo così dimostrato che lo stato squeezed caratterizzato dal valore α del parametro complesso è un autostato dell'operatore di distruzione $\hat{a}$, con autovalore α.

Prendiamo adesso un altro stato coerente $|\beta\rangle$ e applichiamo a questo stato l'operatore $\hat{D}^{-1}(\alpha)\,\hat{a}\,\hat{D}(\alpha)$:

$$\begin{aligned} &\hat{D}^{-1}(\alpha)\,\hat{a}\,\hat{D}(\alpha)\,|\beta\rangle \;=\; (\hat{a} \,+\, \alpha)\;|\beta\rangle \\ &\hat{D}^{-1}(\alpha)\,\hat{a}\,\hat{D}(\alpha)\,|\beta\rangle \;=\; \hat{a}\,|\beta\rangle \,+\, \alpha\,|\beta\rangle \end{aligned}$$

ma abbiamo appena dimostrato che $\hat{a}\,|\beta\rangle \;=\; \beta\,|\beta\rangle$, da cui

$$\begin{aligned} &\hat{D}^{-1}(\alpha)\,\hat{a}\,\hat{D}(\alpha)\,|\beta\rangle \;=\; \beta\,|\beta\rangle \,+\, \alpha\,|\beta\rangle \\ &\hat{D}^{-1}(\alpha)\,\hat{a}\,\hat{D}(\alpha)\,|\beta\rangle \;=\; (\beta \,+\, \alpha)\;|\beta\rangle \end{aligned}$$

da cui, applicando ad ambo i membri l'operatore di spostamento con parametro α :

$$\hat{D}\,\hat{D}^{-1}(\alpha)\,\hat{a}\,\hat{D}(\alpha)\,|\beta\rangle \;=\; (\beta\,+\,\alpha)\;\hat{D}(\alpha)\,|\beta\rangle$$
$$\hat{a}\left[\hat{D}(\alpha)\,|\beta\rangle\right] \;=\; (\beta\,+\,\alpha)\;\left[\hat{D}(\alpha)\,|\beta\rangle\right]$$

il che significa che *lo stato ottenuto applicando l'operatore di spostamento con parametro α allo stato coerente di parametro β si ottinene un altro stato coerente (infatti è autostato dell'operatore di distruzione), di parametro $(\alpha + \beta)$ (autovalore).*

Questo 'da' conto' del nome dell'operatore : quest'operatore, applicato ad uno stato coerente, ne 'sposta' il parametro.

Questo fatto significa che gli operatori di spostamento, oltre ad essere unitari, formano un *gruppo*.

4.3.2 Proiezione degli stati coerenti sugli stati di Fock

Calcolare la proiezione di uno stato coerente su uno stato di Fock (stato numero) significa calcolare la probabilità che in quello stato coerente ci sia quel numero di fotoni.

L'insieme degli stati numero sono un insieme completo, e dunque possiamo sviluppare lo stato coerente su questa base

Chapter 5

Quantizzazione del campo (III)

Another chapter about second quantization

5.1 Rappresentazione della radiazione elettromagnetica - le quadrature

5.1.1 Formalismo classico

queste note sono prese da [BR04]

Cominciamo con la descrizione della radiazione elettromagnetica, nell'ambito della rappresentazione classica di Maxwell.

Innanzitutto se si risolve l'equazione delle onde si ottiene la seguente forma per le soluzioni:

$$\vec{E}(\vec{r},t) = E_0 \left[\alpha(\vec{r},t)\, e^{i\omega t} + \alpha^*(\vec{r},t)\, e^{-i\omega t}\right] \hat{p}(\vec{r},t). \tag{5.1}$$

dove $\alpha(\vec{r},t)$ è detta *ampiezza complessa*, $\vec{k}$ è il vettore d'onda, e $\hat{p}$ è il versore di polarizzazione (proiezione del vettore campo elettrico nel piano perpendicolare alla direzione di propagazione), e l'ampiezza complessa è:

$$\alpha(\vec{r},t) = \alpha_0(\vec{r},t)e^{i\phi(\vec{r},t)} \tag{5.2}$$

Esplicitando l'ampiezza complessa:

$$\vec{E}(\vec{r},t) = E_0\alpha_0(\vec{r},t)\left[e^{i[\phi(\vec{r},t)+\omega t]} + e^{-i[\phi(\vec{r},t)+\omega t]}\right] \hat{p}(\vec{r},t) \tag{5.3}$$

La fase $\phi(\vec{r},t)$ descrive la *geometria del fronte d'onda*, nonché la *differenza di fase* (dell'andamento oscillante) rispetto ad una certa onda di riferimento.

Nel caso di *onda piana monocromatica* si ha la familiare forma:

$$\vec{E}(\vec{r},t) = E(\vec{r},t)\,[e^{i(kz-\omega t)} + e^{-i(kz-\omega t)}]\,\hat{p}(\vec{r},t) \tag{5.4}$$

dove z è la direzione di propagazione e $k = \frac{2\pi}{\lambda}$ è il numero d'onda.

Quadrature in formalismo classico

Vogliamo ora passare ad un'altra notazione che tenga separati i due significati della fase (geometria dei fronti d'onda e fase della parte oscillante). (ma qua non ho capito se la riscrittura con le quadrature è per ottenere questa separazione dei "significati" della fase!)

Riscriviamo la (5.1) :

$$\vec{E}(\vec{r},t) = E_0\,[\alpha(\vec{r},t)\ e^{i\omega t} + \alpha^*(\vec{r},t)\ e^{-i\omega t}]\,\hat{p}(\vec{r},t) \tag{5.5}$$

e applicando le formule di Eulero

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{5.6a}$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \tag{5.6b}$$

si ha:

$$\vec{E}(\vec{r},t) = E_0[\alpha(\vec{r},t)\ e^{i\omega t} + \alpha^*(\vec{r},t)\ e^{-i\omega t}]\,\hat{p}(\vec{r},t) \tag{5.7a}$$

$$= E_0[\alpha(\vec{r},t)\ (\cos\omega t + i\sin\omega t) + \alpha^*(\vec{r},t)\ (\cos\omega t - i\sin\omega t)]\,\hat{p}(\vec{r},t) \tag{5.7b}$$

$$= E_0\{\ [\alpha(\vec{r},t) + \alpha^*(\vec{r},t)]\cos\omega t + i[\alpha(\vec{r},t) - \alpha^*(\vec{r},t)]\sin\omega t\ \}\,\hat{p}(\vec{r},t) \tag{5.7c}$$

$$= E_0\,[X_1(\vec{r},t)\cos\omega t + X_2(\vec{r},t)\sin\omega t]\ \hat{p}(\vec{r},t) \tag{5.7d}$$

dove abbiamo definito le quadrature:

$$X_1 \equiv \alpha(\vec{r},t) + \alpha^*(\vec{r},t) \tag{5.8a}$$

$$X_2 \equiv i[\alpha(\vec{r},t) - \alpha^*(\vec{r},t)] \tag{5.8b}$$

D'altra parte, guardando alla definizione di ampiezza complessa 5.2, possiamo scrivere

$$X_1 = 2\alpha_0(\vec{r}, t)\cos\phi(\vec{r}, t) \tag{5.9a}$$

$$X_2 = 2i\alpha_0(\vec{r}, t)\sin(\vec{r}, t) \tag{5.9b}$$

Il "significato" delle quadrature si ha se immaginiamo di rappresentare il campo dell'onda sul piano dei fasori o *piano di Gauss*, che viene usato per rappresentare su un piano complesso la parte a fattore della "semplice rotazione costante" descritta dai termini in ωt del campo, detta appunto "fasore". Le quadrature dunque sono le due componenti cartesiane del fasore del campo dell'onda. La parte dipendente da ωt descrive poi una "rotazione costante", che nella rappresentazione dei fasori viene esclusa. Notiamo che questo fasore in generale dipende dal tempo, mentre invece nel caso di onda piana monocromatica non dipende dal tempo, e ha un andamento oscillante in funzione della posizione. Se sono presenti più onde, queste faranno interferenza. A tempo e posizione fissata, il fasore dell'onda risultante dall'interferenza è la somma vettoriale dei fasori.

5.1.2 Formalismo quantistico

Richiamiamo brevemente la quantizzazione del campo elettromagnetico. Si comincia con lo scrivere l'energia elettromagnetica (per semplicità riferiamoci ad una radiazione contenuta in una cavità).

[...]

seconda quantizzazione (vedi anche file di teoria)

- campo in una cavità (parallelepipedica)

- modi di Slater

- hamiltoniana

- si riconosce la forma dell'hamiltoniana come somma di hamiltoniane di oscillatori armonici

- si descrive il sistema come un insieme di oscillatori armonici

[...]

[. . .]

rappresentazione con gli stati numero

Dunque in questo formalismo uno stato del sistema "campo elettromagnetico" si può rappresentare tramite un elemento dello spazio di Fock, (*stati di Fock* o *stati numero*). Tali stati si possono definire come gli autostati dell'operatore numero, ovvero, come prodotto di un (infinito) numero di tali autostati, in quanto in linea di principio i livelli energetici degli oscillatori armonici sono in numero infinito (sebbene in generale solo un numero finito ha numero di occupazione diverso da zero). Questa rappresentazione è in qualche modo antiintuitiva: abbiamo detto di descrivere il campo elettromagnetico come un insieme di oscillatori armonici, uno per ogni modo elettromagnetico (vedi modi di Slater). A sua volta ognuno di questi oscillatori può essere in uno degli infiniti livelli energetici discreti (quantizzazione). La rappresentazione degli stati numero, anziché dire che il il k-esimo oscillatore (k-esimo modo) è nel livello eccitato l, dice che nel livello l-esimo si trovano n-oscillatori (n-modi).

Questa rappresentazione degli stati del campo, ha il vantaggio di derivare direttamente dalla teoria, cioè dal modo in cui si "quantizza" il campo. Tuttavia è difficile rappresentare in questo formalismo uno stato con fase fissata.

rappresentazione con gli stati coerenti

(Per quanto segue sugli stati coerenti ho preso dai miei appunti di ottica quantistica) Una rappresentazione degli stati del campo EM alternativa alla rappresentazione con gli stati di Fock, è la rappresentazione con gli stati coerenti. Il fatto di passare da una rappresentazione ad un'altra può essere visto come un cambio di riferimento dello spazio di Fock (controllare...). L'insieme degli stati coerenti è un insieme completo, ma non è ortogonale.

Gli stati coerenti si possono definire come autostati dell'operatore di annichilazione $\hat{a}^\dagger$:

$$\hat{a}^\dagger|\alpha\rangle = \alpha|\alpha\rangle \tag{5.10}$$

α, oltre ad essere l'autovalore corrispondente, è un parametro (complesso, $\alpha \in \mathbb{C}; \alpha = |\alpha|e^{\phi}$) che caratterizza lo stato coerente.

Un modo alternativo di introdurre gli stati coerenti è quello di definirli come gli stati che si ottengono applicando l'operatore di spostamento

$$\hat{D}(\alpha) \equiv e^{-\alpha^* \hat{a} + \alpha\, \hat{a}^\dagger}. \tag{5.11}$$

Infine, si può dimostrare che gli stati coerenti sono gli stati del campo EM in presenza di una corrente classica (antenna).

Se si misura il vettore campo elettrico di uno stato coerente $|\alpha\rangle$, il valore di aspettazione è una funzione oscillante (sinusoidale o cosinusoidale) del tempo, la cui ampiezza è data dal modulo di α e la cui fase è data dalla fase di α.

Quadrature in formalismo quantistico

È possibile introdurre le quadrature facendole "discendere" direttamente dalla quantizzazione del campo.

Se la seconda quantizzazione consiste nel "promuovere ad operatori" le due variabili 'posizione' e 'impulso' che compaiono nell'hamiltoniana del campo elettromagnetico, si possono fare delle trasformazioni (cambi di variabili nello spazio delle fasi) e passare ad altre coppie di operatori coniugati: op di creazione e distruzione, oppure i nostri due operatori di quadratura:

$$\hat{X}_\theta \equiv \frac{1}{2}\left[\hat{a}\, e^{-i\,\theta} + \hat{a}^\dagger e^{i\,\theta}\right]$$

$$\hat{X}_{\theta+\frac{\pi}{2}} = \frac{1}{2}\left[\hat{a}\, e^{-i\left(\theta+\frac{\pi}{2}\right)} \,+\, \hat{a}^{\dagger} e^{i\left(\theta+\frac{\pi}{2}\right)}\right]$$

$$\hat{X}_{\theta+\frac{\pi}{2}} = \frac{1}{2}\left[\hat{a}\, e^{-i\,\theta}\, e^{-i\,\frac{\pi}{2}} \,+\, \hat{a}^{\dagger} e^{i\,\theta}\, e^{i\,\frac{\pi}{2}}\right]$$

$$\hat{X}_{\theta+\frac{\pi}{2}} = \frac{1}{2}\left[\hat{a}\, e^{-i\,\theta}\,(-i) \,+\, \hat{a}^{\dagger} e^{i\,\theta}\, i\right]$$

$$\hat{X}_{\theta+\frac{\pi}{2}} = \frac{1}{2}\,\frac{1}{i}\left[\hat{a}\, e^{-i\,\theta}\, i\,(-i) \,+\, \hat{a}^{\dagger} e^{i\,\theta}\, i\, i\right]$$

$$\hat{X}_{\theta+\frac{\pi}{2}} = \frac{1}{2i}\,\frac{1}{i}\left[\hat{a}\, e^{-i\,\theta} \,-\, \hat{a}^{\dagger} e^{i\,\theta}\right]$$

in definitiva

$$\begin{cases} \hat{X}_{\theta} & = & \frac{1}{2}\left[\hat{a}\, e^{-i\,\theta} + \hat{a}^{\dagger} e^{i\,\theta}\right] \\ \hat{X}_{\theta+\frac{\pi}{2}} & = & \frac{1}{2i}\left[\hat{a}\, e^{-i\,\theta} - \hat{a}^{\dagger} e^{i\,\theta}\right] \end{cases} \tag{5.12}$$

5.2 Rappresentazioni in meccanica quantistica

5.2.1 Rappresentazione di Shrödinger

5.2.2 Rappresentazione di Heisemberg

5.3 Il beam splitter

Seguendo le diverse possibili descrizioni della radiazione, anche la fisica del beam splitter la si può descrivere in formalismo classico (ottica ondulatoria), in formalismo quantistico con la rappresentazione degli stati di Fock (fotoni, rappresentazione corpuscolare) e in formalismo quantistico con la rappresentazione degli stati coerenti (rappresentazione ondulatoria).

5.3.1 Descrizione classica

Consideriamo un fascio incidente, che nel formalismo classico è descritto da **??**:

$$E_{in}\, e^{i\,\phi_{in}} \tag{5.13}$$

questo fascio incidente originerà un fascio trasmesso ed un fascio riflesso, rappresentati rispettivamente da

$$E_t\, e^{i\,\phi_t} \tag{5.14}$$

e

$$E_r\, e^{i\,\phi_r}. \tag{5.15}$$

Ciò posto, è possibile caratterizzare il beam splitter per mezzo di un *coefficiente di riflessione* ϵ che descrive le intensità (moduli quadri) dei fasci trasmesso e riflesso in funzione di quella del fascio incidente:

$$\begin{cases} |E_r|^2 &= \epsilon|E_{in}|^2 \\ |E_t|^2 &= (1-\epsilon)|E_{in}|^2. \end{cases} \tag{5.16}$$

Una descrizione più generalizzata prevede un secondo fascio in ingresso dalla seconda porta:

$$E_u\, e^{i\,\phi_u}. \tag{5.17}$$

Per questo fascio il coefficiente di riflessione è $(1-\epsilon)$, mentre ϵ è il coefficiente di trasmissione.

Assumendo che il beam splitter non abbia perdite o assorbimenti, l'energia totale si conserverà:

$$|E_{in}|^2+|E_u|^2=|E_r|^2+|E_t|^2. \tag{5.18}$$

Questo riguardo alle intensità. Riguardo alle ampiezze il discorso è più complicato, perché bisogna considerare l'effetto del beam splitter sulle fasi. Se stabiliamo arbitrariamente che le fasi dei due fasci incidenti siano nulle, cioè $\phi_{in} = \phi_u = 0$. [...]

Part II

English notes

Chapter 6

Electromagnetic field quantization

6.1 Classical field

We want to have a quantum description of the electromagnetic field, and develop the formalism to work with it, in particular to describe interference phenomena. Let's start with a classical description, and consider the *eigenmodes* of the electromagnetic field. In a classical description those are just *modes*, i.e. "modes of oscillation" of the fields; it has to be possible to describe any oscillation as a linear combination of these "basis modes". To fix the ideas we can think about the (eigen)modes of the field in a parallelepipedic cavity, which we know to be *plane waves*. This is just to fix ideas: in different configurations (boundary conditions) we will have different eigenmodes. Generally speaking these fundamental modes are known as the *Slater modes*:

$$\left\{ \begin{array}{rcl} \vec{E}\,(\vec{r},t) & = & -\frac{1}{\sqrt{\epsilon}} \sum_k p_k(t)\, \vec{E}_k\,(\vec{r}) \\ \vec{H}\,(\vec{r},t) & = & \frac{1}{\sqrt{\mu}} \sum_k \omega_k\, q_k(t)\, \vec{H}_k\,(\vec{r})\,. \end{array} \right. \tag{6.1}$$

We can consider the $p_k(t)$s and $q_k(t)$s that appear in this formula simply as (time dependent) coefficients, while k is a cumulative index, which incorporates all the indices needed to individuate a single eigenmode (frequency / wave vector, polarization). It is worth to note that in the $p_k(t)$ and $q_k(t)$ coefficients appears the *time dependence*, while the eigenmodes $\vec{E}_k\,(\vec{r})$ and $\vec{H}_k\,(\vec{r})$ carry the information about the spatial dependence.

We can assume for simplicity that the eigenmodes are orthonormal:

$$\begin{cases} \int_{\substack{\text{vol.}\\ \text{cav.}}} \vec{E}_k(\vec{r}) \cdot \vec{E}_l(\vec{r})\, dV &= \delta_{k\,l} \\ \int_{\substack{\text{vol.}\\ \text{cav.}}} \vec{H}_k(\vec{r}) \cdot \vec{H}_l(\vec{r})\, dV &= \delta_{k\,l}. \end{cases} \tag{6.2}$$

6.2 Harmonic oscillator Hamiltonian

[...] in classical formalism we write the energy of the electromagnetic field. Then we recognize the hamiltonian of harmonic oscillators. [...]

6.3 Promotion

At this stage we "promote" the p_n and q_n coefficients to operators:

$$p_n(t) \rightarrow \hat{p}_n(t) \tag{6.3}$$

$$q_n(t) \rightarrow \hat{q}_n(t). \tag{6.4}$$

About these operators, we don't have at this stage an explicit *representation*; we now introduce two "auxiliary operators" $\hat{a}$ and $\hat{a}^\dagger$, and we will express $\hat{q}_n(t)$ and $\hat{p}_n(t)$ in terms of these auxiliary operators:

$$\begin{cases} \hat{p}_n(t) &= \frac{\sqrt{2\hbar\omega_a}}{2}\left[\hat{a}_n e^{-i\,\omega_n t} + \hat{a}_n^\dagger e^{i\,\omega_n t}\right] \\ \hat{q}_n(t) &= \frac{\sqrt{2\hbar\omega_a}}{2}\left[\hat{a}_n e^{-i\,\omega_n t} - \hat{a}_n^\dagger e^{i\,\omega_n t}\right] \end{cases} \tag{6.5}$$

Let's stress that here the time dependence is in the form of an imaginary exponential, because this is a particular case, related with the choice of the Slater's modes as *plane waves*. With different boundary conditions we have different Slater's modes, and different time dependence.

If we put these explicit expressions of the Slater's modes in the expansion of the (electric) field we have:

$$\hat{E}(\vec{r},t) = \frac{1}{\sqrt{\epsilon}} \sum_n \left[E^*(\vec{r})\, \hat{a}_n^\dagger\, e^{i\,\omega_n t} + E(\vec{r})\, \hat{a}_n\, e^{-i\,\omega_n t}\right] \tag{6.6}$$

$$\hat{E}(\vec{r},t) = \hat{E}^{(-)}(\vec{r},t) \,+\, \hat{E}^{(+)}(\vec{r},t) \tag{6.7}$$

where we have defined the two *hermitian conjugate* (h.c.) operators

$$\begin{cases} \hat{E}^{(+)}(\vec{r},t) = \frac{1}{\sqrt{\epsilon}} \sum_n E(\vec{r})\, \hat{a}_n\, e^{-i\,\omega_n t} \\ \hat{E}^{(-)}(\vec{r},t) = \frac{1}{\sqrt{\epsilon}} \sum_n E^*(\vec{r})\, \hat{a}_n^\dagger\, e^{i\,\omega_n t}. \end{cases} \tag{6.8}$$

The first of those two h.c. operators $\hat{E}^{(+)}(\vec{r},t)$ represent the "positive frequency part", and the second, $\hat{E}^{(-)}(\vec{r},t)$ represent the "negative frequency part".

This said, we can write the intensity of the field as an *expectation value*. We are interested in the positive frequency part (since in our slice of the Minkowski space the time moves forward):

$$I = \langle\psi|\hat{E}^{(-)}(\vec{r},t)\, \hat{E}^{(+)}(\vec{r},t)|\psi\rangle \tag{6.9}$$

expliciting:

$$I = \frac{1}{\epsilon} \sum_{n,m} E_n^*(\vec{r}) E_m(\vec{r})\, e^{-i\,(\omega_m - \omega_n)\,t}\, \langle\psi|\hat{a}_n^\dagger\, \hat{a}_m|\psi\rangle \tag{6.10}$$

If our state $|\psi\rangle$ is a "single photon" state:

$$|\psi\rangle = |1\rangle_q \tag{6.11}$$

which can be written as

$$|1\rangle_q = \hat{a}_q^\dagger|0\rangle \tag{6.12}$$

then the *expectation value* of the intensity on this state is:

$$\frac{1}{\epsilon} \sum_{n,m} E_n^*(\vec{r})\, E_m(\vec{r})\, e^{-i\,(\omega_m - \omega_n)\,t}\; \delta_{nq}\, \delta_{mq}\, \langle 0|0\rangle = \frac{1}{\epsilon} |E_q(\vec{r})|^2. \tag{6.13}$$

6.4 Creation and annihilation operators

[. . .] definition of the creation and annihilation operators, their analytic expression, their properties (action on single photon states)

6.4.1 Commutation relations

$$\begin{cases} [\hat{a}_n, \hat{a}_m] = \left[\hat{a}_n^\dagger, \hat{a}_m^\dagger\right] = 0 \\ \left[\hat{a}_n, \hat{a}_m^\dagger\right] = \left[\hat{a}_n^\dagger, \hat{a}_m\right] = \delta_{n\,m} \end{cases} \tag{6.14}$$

[. . .]

6.5 Fock space and states of the e.m. field

The state of the e.m. field has to be represented by a vector in some Hilbert space, so here we want to introduce this suitable space.

6.5.1 Single Hilbert space

For single particles like electrons, we are used to use the Hilbert space of *complex wave functions* $|\psi\rangle \in \mathcal{H}$.

When we have to represent a system with more than one particle, let's say N particles (or in general "sub-systems") we use the tensor product:

$$|\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots |\psi_N\rangle \in \bigotimes_{i=1}^{N} \mathcal{H} = \mathcal{H}^{\otimes N} \tag{6.15}$$

(then we have to make it symmetric or antisymmetric depending whether the particles are bosons or fermions). For details about *tensor product* and *direct sum*, refer to appendix E.

6.5.2 Single mode Fock space

When we deal with the e.m. field, we can create or annihilate particles (photons), so that the total number of particles is *not* constant. Since the Hilbert space we have introduced can represent only systems with a fixed number N of particles (tensor proucts), the state of the system has to "jump" from a Hilbert space to another. So we have to build a more suitable space, to describe the states of our system. We do so "expanding" the idea of the Hilbert space (which has a fixed number of particles) by creating a space named *Fock space*, which is the (infinite) direct sum of different Hilbert spaces, each of them with a different fixed number of particles:

$$\mathcal{F} = \bigoplus_{N=0}^{\infty} \mathcal{H}^{\otimes N}. \tag{6.16}$$

Again, for details about *tensor product* and *direct sum*, refer to appendix E.

A Fock space is mathematically still a Hilbert space, but with an undefined number of particles.

6.5.3 Multimode Fock space

Up to now, we have not taken into account the different modes of the e.m. field: we "jump" from one Hilbert space with N particles to the one with $N+1$ or to the one with $N-1$ with the creation or annihilation operators, but we are always dealing with the same *electromagnetic mode*, after which these operators are defined. So we can name this Fock space we just defined as a *single-mode Fock space*.

We want now a "bigger" *multi-mode Fock space*.

If we consider the easier situation of a confined e.m. field, where the modes are discrete, we can define this multi-mode Fock space as the tensor product of infinite "single mode" Fock spaces:

$$\begin{aligned}\mathcal{F}^{\text{mul}} &\equiv \bigotimes_{m=1}^{\infty} \mathcal{F}_m \\ &= \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \ldots \mathcal{F}_m \otimes \ldots\end{aligned} \tag{6.17}$$

where each *subscript* refers to a *different electromagnetic mode*.

Chapter 7

Time-bin operators formalism

7.1 Introduction

In the previous sections we have seen that creation and annihilation operators (6.4) deal with single photons, i.e. single excitations of the electromagnetic field. But the single photon state that $\hat{a}^{\dagger}$ creates is completely delocalized (plane wave), and so is "unrealistic" (is displaced over an infinite region of space, so has infinite energy, etc..). The *time-bins* formalism develops the formal tools to describe "real" single photons states.

This is based on the review work in [KNR+07], in particular on section IV (Realistic Optical Components and Their Errors), subsection B (Photon sources). (Add a reference to the book by Pieter Kok and Brandon Lowet)

7.2 Mode functions

In this section we discuss the single modes and the possible sets of single modes used as basis to write any other mode.

7.2.1 General localized single-photon states

It is possible to consider an abstract set of modes, and in this case each mode is identified just by a label j, with no specific physical meaning.

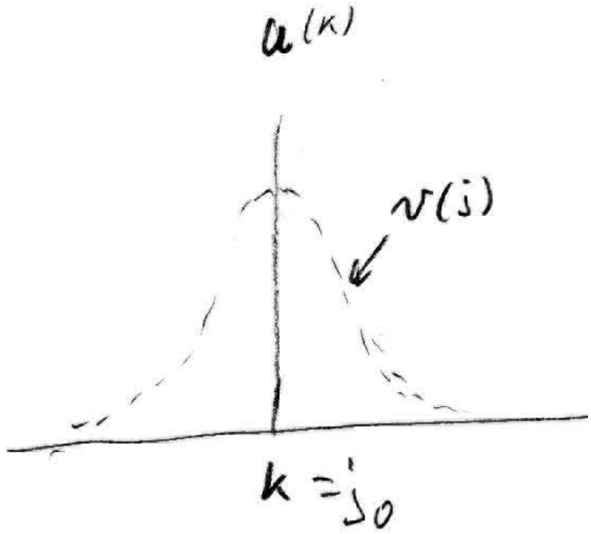

Figure 7.1: mode functions represented in the wave number space

We can build a single photon state that is the superposition of single-mode single photons:

$$|1; f\rangle = \sum_{n=1}^{\infty} f_n \hat{a}^{\dagger}_{\omega_n} |0\rangle \tag{7.1}$$

One possible goal of writing such a linear combination, is to obtain a localized single-photon state of the electromagnetic field. The f_n are coefficients that specify the "packet" (single mode amplitudes), have to fulfill the normalization relation $\sum_n |f_n|^2 = 1$. Is owrth pointing out that the vacuum state is the product of the "monochromatic vacuum states": $|0\rangle = \bigotimes_n |0\rangle_{\omega_n}$.

7.2.2 Plane waves

It is possible to show that the plane waves:

$$u(\vec{k}; \vec{r}, t) = \frac{e^{i(\vec{k}\cdot\vec{r} \, - \, \omega_{\vec{k}}\, t)}}{\sqrt{(2\pi)^3 \, 2\,\omega_{\vec{k}}}} \tag{7.2}$$

Are solutions of the wave equations (derived from the Maxwell's equations) in a cavity with flat parallel walls. In other words the plane waves can be used as *Slater modes* (6.1), to describe the radiation in such cavity.

In the following we also consider a generic set of infinite modes, labeled by n, where the range will be $0 \leq n < \infty$, or $-\infty < n < +\infty$.

In the specific case of the plane waves, this label can be related to the wave vector $\vec{k}$ *and* the frequency (pulsation) ω.

If $\hat{a}_n$ are plane waves modes, and we use as coefficients the exponential functions $e^{-i\omega_n t}$, we can look at the linear combination as a Fourier transform:

$$\hat{a}(t) = \sum_{n=1}^{\infty} \hat{a}_n e^{-i\omega_n t} \tag{7.3}$$

This can be seen as a Fourier transform, and so the inverse relation will be the anti-transform:

$$\hat{a}_{\omega_n} = \sum_{t=1}^{\infty} \hat{a}(t) e^{i\omega_n t} \tag{7.4}$$

where in the inverse transform we have "discretized" the time.

7.2.3 time-bins operators

Let's now introduce a different set of creation operators (with which to build the "localized photon") defined as functions of the single-mode creation operators introduced in (7.4):

$$\boxed{\hat{b}_\mu \ := \ \frac{1}{\sqrt{N}} \sum_{m=1}^{N} e^{-i\tau\mu m} \ \hat{a}_m} \ . \tag{7.5}$$

where we have set $\tau \equiv \frac{2\pi}{N}$ so that $(\tau\mu)$ goes from $\frac{2\pi}{N}$ to 2π. The inverse relation is:

$$\hat{a}_m \ = \ \frac{1}{\sqrt{N}} \sum_{\mu=1}^{N} e^{i\tau\mu m} \ \hat{b}_\mu . \tag{7.6}$$

7.2.4 $\hat{a}(t)$ as a function of the time-bin operators

If we insert formula (7.5) (which defines the time-bins operators as function of the standard creation operators) in formula (7.3) (which expresses the time dependent creation operator $\hat{a}(t)$) we obtain a new expression of $\hat{a}(t)$ (for simplicity we replace $\omega_n = n$):

$$\begin{aligned}\hat{a}(t) &= \sum_{n=1}^{\infty} \hat{a}_n e^{-i\,n\,t} \\ &= \sum_{n=1}^{\infty} \overbrace{\frac{1}{\sqrt{N}} \sum_{\mu=1}^{N} e^{i\,\tau\mu\,n}\,\hat{b}_\mu}\; e^{-i\,n\,t} \\ &= \sum_{\mu=1}^{N} g_\mu(t)\,\hat{b}_\mu\end{aligned}$$

where we have defined:

$$g_\mu(t) := \frac{1}{\sqrt{N}} \frac{e^{i\,(\tau\mu - t)}}{1 - e^{i\,(\tau\mu - t)}} \tag{7.7}$$

For the details of this derivation see appendix A

7.3 Summary and comments

Generally speaking, to (classically) describe the electromagnetic field, we choose a complete set of wave functions (Slater modes), and we describe any state of the field as a linear combination of this "basis set". We have several possible choices of the basis, the plane waves being one choice.

Once we quantize the field, we have also seen that with each electromagnetic mode we can associate a function (mode function) that describes the "shape" of the mode, *and* a creation (and annihilation) operator to "generate" the states.

This is the most general structure.

Then we can change the set of mode functions that we use as a basis. As an example change from plane waves to the gaussian modes. Another example is a change from the time domain to the frequency domain.

7.3.1 Time & frequency

Is worth to point out that we can distinguish between *spatial modes* and *frequency modes*. In other terms we can label the Slater modes in (6.1) with wave vectors, so to have *spatial modes* or with frequencies, so to have *frequency modes*. This distinction is just for the sake of completeness, but is not so crucial, since there is a relation between the wave vector and the frequency, called *dispersion relation*:

$$\vec{k} = \vec{k}(\omega). \tag{7.8}$$

In free space this relation is a simple linear relation, and if we are confined in a cavity, the discrete wave vectors and frequencies will have the following relation:

$$\vec{k_n}(\omega_n) = \frac{\omega_n}{c} \tag{7.9}$$

where c is the speed of light, and for simplicity we have considered only one spatial dimension. The explicit expression of wave vector and frequency, as function of the cavity width L are:

$$\begin{cases} k_n = \frac{n\pi}{L} \\ \omega_n = c\frac{n\pi}{L}. \end{cases} \tag{7.10}$$

7.3.2 Physical intuition on time-bins

Here are some physical intuitions on the time-bin operators.

On operators: time domain

We already gave an intuitive meaning to the $\hat{a}(t)$ operators in (7.3) as the "Fourier transform" of the $\hat{a}_n$ operators: the $\hat{a}_n$ are defined in the frequency domain, and the $\hat{a}(t)$ in the time domain, the Fourier transform being a way to switch from one domain to the other.

Since we wanted a simpler situation, we decided to limit the space domain of the electromagnetic field, so that the set of allowed frequencies is discrete, and when we Fourier-transform, we have a *periodic time domain*, with period $[0, 2\pi)$.

From the Fourier transform properties we know that if we make the allowed frequencies *more and more dense*, the period of the time-domain transform becomes *larger and larger*. In the limit of continuous frequencies, the period will be the whole real (time) axis. So, when we look at the $[0, 2\pi)$ period, we could see it as "the whole real axis, that has been shrunk because we chose discrete frequencies". Moreover, when in the following we will find that $n(t)$ is periodic within $[0, 2\pi)$, we have to keep in mind that in the limit of continuous frequencies, it will be "periodic on the whole time axis" (meaning that will not be periodic, i.e., it will have a single period, long as the whole real axis).

On states: localized single photons vs "time-bin localized" single photons

We have seen that the single photon state $|1; f\rangle$ in (7.1) is a "localized single photon". We obtain this locality starting with "single-mode single photon" states (plane wave, totally delocalized in the whole space and time), and building a "wave packet" adding many of them.

We can do this with any Slater modes, *e.g.* we can do this with the *plane waves*. But although we have localized the photon, so far we can localize it only in the origin.

With the time-bin formalism, we introduce the quantity $\tau \equiv \frac{2\pi}{N}$, that is "the N-th part of the time period", so when we write $\tau\mu$ with $\mu \in \mathbb{N}$, we have discretized our time domain. In this way, for each choice of $\mu \leq N$ we identify a different "time-bin", and translate the function from $t = 0$ to $t = \mu\tau$. This gives us the freedom to define a photon localized in whatever time-bin we like. Another way to see this is to look at the definition of the time-bin operator (7.5) as a wave packet centered on $t = \mu\tau$. So each "creation time-bin operator" $\hat{b}^{\dagger}_{\mu}$ creates a photon localized in the time-bin $t = \mu\tau = \frac{\mu}{N}2\pi$. Is worth remembering that the linear relationship between the plane wave modes $\hat{a}_m$ and the time-bin modes $\hat{b}_{\nu}$ is realised by a unitary transformation that *does not change particle number*. Another way to state this is to say that there is no mixing of creation and annihilation operators.

To see this more clearly, let's compare the "localized single photon" state $|1; f\rangle$ in (7.1)

$$|1;f\rangle = \sum_{m=1}^{\infty} f_m \; \hat{a}_m^\dagger \; |0\rangle$$

with a single photon state built using a time-bin operator:

$$\begin{aligned} |1_\mu\rangle &= \sum_{\mu=1}^{\infty} \alpha_\mu \; \hat{b}_\mu^\dagger \; |0\rangle \\ &= \sum_{\mu=1}^{\infty} \alpha_\mu \left[\sum_{m=1}^{N} e^{i\tau\mu m} \; \hat{a}_m^\dagger \right] |0\rangle \end{aligned}$$

where the role of α_m in the latter is the same of f_m in the previous, i.e. coefficients used in the sum to form the "packet". Note the convention used: a greek letter subscript to the occupation number will mean that the photon has been created with a time-bin operator.

7.4 Photon numbers

In this section we want to compute the expectation values of the number operator on various states, including plane waves single photon states, time-bin single photon states, and time-bin two photon states.

7.4.1 Expectation value of the number operator on a single photon time-bin state

The expectation value of the "number operator" $\hat{n}(t) = \hat{a}^\dagger(t)\hat{a}(t)$ on this state is:

$$\langle 1; f|\hat{n}|1; f\rangle = \sum_{n=1}^{\infty} |f_n\, e^{-i\,\omega_n t}|^2. \tag{7.11}$$

For the detailed derivation of this result see appendix G.2.1 on page 101.

7.4.2 Photon number on single-photon time-bin state

Let's consider a state containing one single photon, but built with the time-bin creation operator $\hat{b}^\dagger_\mu$:

$$|1_\mu\rangle = \hat{b}^\dagger_\mu|0\rangle. \tag{7.12}$$

This state has no special physical meaning. Only a linear combination of these states (packet) has the meaning of single photon localized in a time-bin. The expectation value of the number operator on such state is:

$$\langle 1_\mu|\hat{n}(t)|1_\mu\rangle = |g_\mu(t)|^2. \tag{7.13}$$

For the derivation of this result see appendix G.2.5 on page 106.

7.4.3 Photon number on two-photon time-bin state

First, let's build a state with two localized photons in two different time-bins:

$$|1_\mu, 1_\nu\rangle = \hat{b}^\dagger_\mu \hat{b}^\dagger_\nu|0, 0\rangle \qquad \mu \neq \nu. \tag{7.14}$$

The expectation value of the number operator (mean photon number) on such a state is:

$$\langle 1_\mu, 1_\nu | \hat{n}(t) | 1_\mu, 1_\nu \rangle = |g_\mu(t)|^2 + |g_\nu(t)|^2. \tag{7.15}$$

The calculations for this result are in appendix G.2.6 on page 107.

7.4.4 Photon number on single-photon time-bin-packet state

Let's see now the expectation value of the number operator on a state built with a packet (linear combination) of time-bin operators. This will be a single photon state localized in a time-bin:

$$|\tilde{1}; \alpha\rangle = \sum_\mu \alpha_\mu \, \hat{b}^\dagger_\mu |0\rangle. \tag{7.16}$$

On this state, the expectation value of the number operator is:

$$\langle \tilde{1}; \alpha | \hat{n}(t) | \tilde{1}; \alpha \rangle = \sum_{\mu\,\nu} \alpha^*_\mu \, \alpha_\nu \, g^*_\mu \, g_\nu \tag{7.17}$$

The calculations for this result are in appendix G.2.7 on page 109.

7.4.5 Photon number on two photon time-bin-packet state

Let's see now the expectation value of the number operator on a state built with a packet (linear combination) of time-bin operators. This will be a single photon state localized in a time-bin:

$$|\tilde{2}; \alpha, \beta\rangle = |\tilde{1}; \alpha\rangle \otimes |\tilde{1}; \beta\rangle \tag{7.18}$$

$$= \left(\sum_\mu \alpha_\mu \, \hat{b}^\dagger_\mu |0\rangle \right) \otimes \left(\sum_\nu \beta_\nu \, \hat{b}^\dagger_\nu |0\rangle \right) \tag{7.19}$$

$$= \sum_{\mu\,\nu} \alpha_\mu \, \beta_\nu \, \hat{b}^\dagger_\mu \, \hat{b}^\dagger_\nu |0\rangle \tag{7.20}$$

$$= \sum_{\mu\,\nu} \alpha_\mu \, \beta_\nu \, |\tilde{1}_\mu, \tilde{1}_\nu\rangle. \tag{7.21}$$

On this state, the expectation value of the number operator is:

$$\langle\tilde{2};\alpha,\beta|\hat{n}(t)|\tilde{2};\alpha,\beta\rangle = [\ldots] \tag{7.22}$$

The calculations for this result are in appendix G.2.8 on page 109.

7.5 Two-time correlation function

Let's now define the "two time correlation function":

$$G^2(T) \equiv \langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+T)\hat{a}(t+T)\hat{a}(t)\rangle; \tag{7.23}$$

the expectation value of this function onto the two time-bin localized photons state (7.14) is:

$$|1_\mu, 1_\nu\rangle = \hat{b}_\mu^\dagger \hat{b}_\nu^\dagger |0,0\rangle \tag{7.24}$$

$$G^2(T) = |g_\mu(t)\; g_\nu(t+T) + g_\nu(t)g_\mu(t+T)|^2. \tag{7.25}$$

See appendix G.2.9 on page 110 for the derivation of this result.

7.5.1 Spatial modes

Up to now we have only dealt with electromagnetic modes that differ only for the frequency or wavelength (or wave number). More generally we can have the case of several different *spatial modes* (i.e. different wave vectors). We can see this as if we have several directions (spatial modes) and within each of these, several frequency modes (with the same spatial mode). This is the case when we deal with photons entering or exiting a beam splitter: for each of the ports we have a different spatial mode, with a different direction of the wave vector. Then for each of the wave vector directions (for each port) we have the frequency or wavelength modes. In this case we need two indices to individuate each Slater mode. For the single mode creation operators (not the time-bin ones) we will use a first letter to label the spatial modes ($a,\ b,\ c...$), and then the usual index for the frequency/wavelength mode ($n,\ m,\ k...$). When we go to the time-bin operators notation, we will keep the letter for the spatial mode, and use the greek letter to individuate the time-bin.

A state with two photons can have the photons in two different time-bin of the same spatial mode (two photons in the same beam splitter port):

$$|1_{a,\mu}, 1_{a,\nu}, 0_b\rangle \tag{7.26}$$

or in two different spatial modes (two different ports):

$$|1_{a,\mu}, 1_{b,\nu}, 0\rangle. \tag{7.27}$$

In such a situation with several spatial modes, the effect of an annihilation operator will be different on these two different two-photon states:

$$\hat{a}_\lambda|1_{a,\mu}, 1_{a,\nu}, 0_b\rangle = \delta_{\lambda\mu}|1_{a,\nu}, 0_b\rangle + \delta_{\lambda\nu}|1_{a,\mu}, 0_b\rangle \tag{7.28}$$

7.6 Hong-Ou-Mandel effect

7.6.1 Beam splitter formalism

In a general *Heisemberg notation*, the action of a Beam Splitter (B.S.) is described as:

$$\begin{pmatrix} \hat{a}^\dagger \\ \hat{b}^\dagger \end{pmatrix} = \begin{pmatrix} \sqrt{\tau} & \sqrt{1-\tau} \\ \sqrt{1-\tau} & -\sqrt{\tau} \end{pmatrix} \begin{pmatrix} \hat{c}^\dagger \\ \hat{d}^\dagger \end{pmatrix} \tag{7.29}$$

where τ is the transmission coefficient of the B.S.. In the case where the B.S. is balanced ($\tau = 0.5$) we have:

$$\begin{pmatrix} \hat{a}^\dagger \\ \hat{b}^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \hat{c}^\dagger \\ \hat{d}^\dagger \end{pmatrix} \tag{7.30}$$

explicitly:

$$\begin{cases} \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{c}^\dagger + \hat{d}^\dagger) \\ \hat{b}^\dagger = \frac{1}{\sqrt{2}}(\hat{c}^\dagger - \hat{d}^\dagger). \end{cases} \tag{7.31}$$

This formalism is subtle to understand. It is as if the matrix acts on the "operators after the B.S.".

But this is just a "formal writing". The rationale of this writing is to write the *old operators* $\hat{a}^{dag}$ and $\hat{b}^{dag}$ as functions of the new ones. Then we can replace these *functions of operators* in the expression of the state *before* the B.S.. In this way it is as if we rewrite the state *before* the B.S., using the relation between the operator before and after, and we obtain the states *after*.

This is slightly counter-intuitive since we know that the evolution is "from" the old operators "to" the new operators. But this 2×2 matrix is not an evolution matrix in the Heisemberg formalism, but just a formal tool to express the relationship between new and old operators when the photons pass through a B.S..

7.6.2 Input state: localized two-photons state

The Hong-Ou-Mandel effect is observed when two (identical) photons enter the B.S.. So let's build such a state, using the time-bin operators: so that the photons will be localized:

$$|\psi_{in}\rangle = \sum_{\mu,\nu=1}^{\infty} \alpha_\mu \beta_\nu \hat{a}^\dagger_\mu \hat{b}^\dagger_\nu |0\rangle = |1_a, 1_b\rangle \tag{7.32}$$

where

$$\hat{a}^\dagger_\mu \hat{b}^\dagger_\nu |0\rangle = |1_{a\mu}, 1_{b\nu}\rangle. \tag{7.33}$$

Here and in the following the operators $\hat{a}^\dagger_\mu$ and $\hat{b}^\dagger_\nu$ are time-bin operators, distinguished by the single mode creation/annihilation operators $\hat{a}_m$ and $\hat{b}_n$ by the *greek subscript*.

If we apply the B.S. matrix to this state we will have the
state at the output of the B.S.:

$$\begin{aligned} |\psi_{out}\rangle &= \frac{1}{2} \sum_{\mu,\nu=1}^{\infty} \alpha_\mu \beta_\nu (\hat{c}^\dagger_\mu + \hat{d}^\dagger_\mu)(\hat{c}^\dagger_\nu - \hat{d}^\dagger_\nu)|0\rangle \\ &= \frac{1}{2} \sum_{\mu,\nu=1}^{\infty} \alpha_\mu \beta_\nu \left[\hat{c}^\dagger_\mu \hat{c}^\dagger_\nu - \hat{c}^\dagger_\mu \hat{d}^\dagger_\nu + \hat{d}^\dagger_\mu \hat{c}^\dagger_\nu - \hat{d}^\dagger_\mu \hat{d}^\dagger_\nu\right] |0\rangle \\ &= \frac{1}{2} \sum_{\mu,\nu=1}^{\infty} \alpha_\mu \beta_\nu \left[|1_{c\mu}, 1_{c\nu}, 0_d\rangle - |1_{c\mu}, 1_{d\nu}\rangle + |1_{c\nu}, 1_{d\mu}\rangle - |0_c, 1_{d\mu}, 1_{d\nu}\rangle\right]. \end{aligned} \tag{7.34}$$

7.7 Two photons interference: the Hong-Ou-Mandel effect

Let's consider the case where the *input* state is $|1,1\rangle_{a,b}$, i.e. two (identical) photons enter the 50/50 b.s. (simultaneously), because it is peculiar.

To use the formalism just developed, let's express this state using the creation operators:

$$|1,1\rangle_{a,b} = \hat{a}^\dagger \hat{b}^\dagger \; |0,0\rangle. \tag{7.35}$$

Then, applying the b.s. matrix formalism, these operators, as a function of the ones after the b.s.:

$$\begin{pmatrix} \hat{a}^\dagger \\ \hat{b}^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \hat{c}^\dagger \\ \hat{d}^\dagger \end{pmatrix} \tag{7.36}$$

explicitly:

$$\begin{cases} \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{c}^\dagger + \hat{d}^\dagger) \\ \hat{b}^\dagger = \frac{1}{\sqrt{2}}(\hat{c}^\dagger - \hat{d}^\dagger) \end{cases} \tag{7.37}$$

so, if we substitute this in the expression of the state before the b.s. we have the state after the b.s.:

$$\begin{aligned} &\left[\frac{1}{\sqrt{2}}(\hat{c}^\dagger + \hat{d}^\dagger)\right] \left[\frac{1}{\sqrt{2}}(\hat{c}^\dagger - \hat{d}^\dagger)\right] |0,0\rangle \\ &\frac{1}{2}\left[\hat{c}^\dagger\hat{c}^\dagger - \hat{c}^\dagger\hat{d}^\dagger + \hat{d}^\dagger\hat{c}^\dagger - \hat{d}^\dagger\hat{d}^\dagger\right] |0,0\rangle \end{aligned}$$

now, we observe that $\hat{c}^\dagger$ and $\hat{d}^\dagger$ act on two different modes of the e.m. field, and so *they commute*. This thing is important, and is worth to stress it, because it has important implications. Since the two operators commute, the calculation carry on like this:

$$\begin{aligned} &\frac{1}{2}\left(\hat{c}^\dagger\hat{c}^\dagger - \hat{c}^\dagger\hat{d}^\dagger + \hat{d}^\dagger\hat{c}^\dagger - \hat{d}^\dagger\hat{d}^\dagger\right) |0,0\rangle \\ &\frac{1}{2}\left(\hat{c}^\dagger\hat{c}^\dagger \underbrace{-\hat{c}^\dagger\hat{d}^\dagger + \hat{c}^\dagger\hat{d}^\dagger} -\hat{d}^\dagger\hat{d}^\dagger\right) |0,0\rangle \\ &\frac{1}{2}\left(\hat{c}^\dagger\hat{c}^\dagger - \hat{d}^\dagger\hat{d}^\dagger\right) |0,0\rangle \\ &\frac{1}{2}\left(\hat{c}^\dagger\hat{c}^\dagger|0,0\rangle - \hat{d}^\dagger\hat{d}^\dagger|0,0\rangle\right) \\ &\frac{1}{2}\left(|2,0\rangle - |0,2\rangle\right) \\ &\frac{|2,0\rangle - |0,2\rangle}{2} \end{aligned}$$

The physical meaning of this result is that if two identical photons enter the b.s., in the same time, the output will be either two photons in one output port or two photons in the other output port. The case of the two photons coming one form each port "cancels out". This effect is known as the *Hong-Ou-Mandel effect* [HOM87]. In fig. 7.2 there is a sketch of the possible paths of the photons; if the photons entering the b.s. are indistinguishable, the first two cases are indistinguishable too, and since in the linear combination they have a different sign, they cancel out.

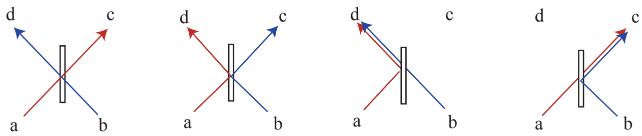

Figure 7.2: Possible paths for the photons

7.7.1 Coincidence function - HOM dip

We want now to evaluate the coincidence function

$$C \equiv \langle \hat{c}^\dagger(t)\hat{d}^\dagger(t+T)\hat{d}(t+T)\hat{c}(t)\rangle \tag{7.38}$$

on this output state of the B.S.. In general we consider the case where the measurement time are not the same on the two output ports of the beam splitter, but are separate by an interval T. The expectation value is:

$$\begin{aligned}\langle\psi_{out}|C|\psi_{out}\rangle &= \\ &= \left(\sum_{\lambda,\varepsilon=1}^{N} |g_\lambda(t)|^2\ |g_\varepsilon(t+T)|^2\right)\left(\frac{1}{2}-\frac{1}{2}\sum_{\mu,\nu=1}^{\infty}\alpha_\mu^*\beta_\nu^*\alpha_\nu\beta_\mu\right).\end{aligned} \tag{7.39}$$

For the detailed calculations see appendix (G.2.11)

Chapter 8

Operative beam-splitter formalism

8.1 Beam Splitter formalism

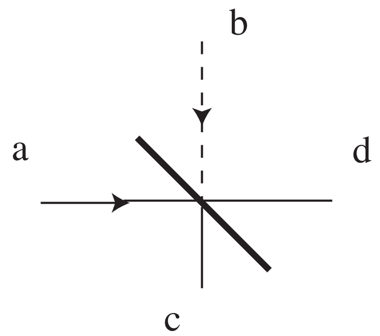

Figure 8.1: beam splitter sketch

To take into account the imperfect balancing of the beam splitter, let's write the Fock states before and after the BS. Before the BS we have $\hat{a}^\dagger|0\rangle = |1_a, 0_b\rangle$. In the Heisemberg formalism we consider the following matrix to represent the effect of the BS on the operators:

$$\begin{pmatrix} \hat{a}^\dagger \\ \hat{b}^\dagger \end{pmatrix} = \begin{pmatrix} \sqrt{r} & \sqrt{t} \\ \sqrt{t} & -\sqrt{r} \end{pmatrix} \begin{pmatrix} \hat{c}^\dagger \\ \hat{d}^\dagger \end{pmatrix} \tag{8.1}$$

where r is bean splitter *reflectance* and t is beam splitter *transmittance*, which are $0 \leq r, t \leq 1$, $r + t = 1$ (ideal loss-less b.s.).

For a balanced BS is $r = t = 0.5$.

To apply the Heisenberg evolution, we have to let the operators evolve, and equation (8.1) is the equation expressing this evolution. Indeed, that equation expresses the relation between the "old" and the "new" operators:

$$\begin{cases} \hat{a}^\dagger &= \sqrt{r}\ \hat{c}^\dagger &+ &\sqrt{t}\ \hat{d}^\dagger \\ \hat{b}^\dagger &= \sqrt{t}\ \hat{c}^\dagger &- &\sqrt{r}\ \hat{d}^\dagger. \end{cases} \tag{8.2}$$

we can also write

$$\begin{cases} \hat{a}^\dagger &\to \sqrt{r}\,\hat{c}^\dagger &+ &\sqrt{t}\ \hat{d}^\dagger \\ \hat{b}^\dagger &\to \sqrt{t}\,\hat{c}^\dagger &- &\sqrt{r}\ \hat{d}^\dagger. \end{cases} \tag{8.3}$$

where the arrow represents the BS (time) evolution. Substituting the expression of $\hat{a}^\dagger$ in the input state we have the output state:

$$\begin{aligned} \hat{a}^\dagger|0\rangle &\to (\sqrt{r}\,\hat{c}^\dagger + \sqrt{t}\,\hat{d}^\dagger)|0\rangle = \\ &= \sqrt{r}\,\hat{c}^\dagger|0\rangle + \sqrt{t}\,\hat{d}^\dagger\,|0\rangle \\ &= \sqrt{r}\,|1_c, 0_d\rangle + \sqrt{t}\ \ |0_c, 1_d\rangle \end{aligned} \tag{8.4}$$

Appendix A

derivation of the time-bin expansion function

Here we report the full computation for the time-bin expansion function presented in section 7.2.4. If we insert formula (7.5) which expresses the time-bins creation operators as function of the standard creation operators, in formula (7.3), which expresses the time dependent creation operator $\hat{a}(t)$, we obtain a new expression of $\hat{a}(t)$ (for simplicity we replace $\omega_n = n$):

$$\begin{aligned}
\hat{a}(t) &= \sum_{n=1}^{\infty} \hat{a}_n e^{-i\,n\,t} \\
&= \sum_{n=1}^{\infty} \overbrace{\frac{1}{\sqrt{N}} \sum_{\mu=1}^{N} e^{i\,\tau\mu\,n}\ \hat{b}_\mu}\ e^{-i\,n\,t} \\
&= \frac{1}{\sqrt{N}} \sum_{n=1}^{\infty} \sum_{\mu=1}^{N} e^{i\,\tau\mu\,n}\, e^{-i\,n\,t}\ \hat{b}_\mu \\
&= \frac{1}{\sqrt{N}} \sum_{\mu=1}^{N} \left[\sum_{n=1}^{\infty} \left[e^{i\,(\tau\mu - t)} \right]^n \right] \hat{b}_\mu
\end{aligned}$$

here we recognize the geometric series $\sum_{n=1}^{\infty} \lambda^n = \frac{\lambda}{1-\lambda}$ so we can substitute the content of the outer brackets

$$\sum_{n=1}^{\infty} \left[e^{i\,(\tau\mu - t)} \right]^n = \frac{e^{i\,(\tau\mu - t)}}{1 - e^{i\,(\tau\mu - t)}}$$
$$= \frac{e^{i\,(\tau\mu - t)}}{1 - e^{i\,(\tau\mu - t)}}$$

so that

$$\hat{a}(t) = \frac{1}{\sqrt{N}} \sum_{\mu=1}^{N} \frac{e^{i\,(\tau\mu - t)}}{1 - e^{i\,(\tau\mu - t)}}\ \hat{b}_\mu$$

and if we define

$$g_\mu(t) \equiv \frac{1}{\sqrt{N}} \frac{e^{i\,(\tau\mu - t)}}{1 - e^{i\,(\tau\mu - t)}} \tag{A.1}$$

we finally have

$$\hat{a}(t) = \sum_{\mu=1}^{N} g_\mu(t)\, \hat{b}_\mu. \tag{A.2}$$

Appendix B

Appendix on the dispersion relations and the group velocity

phase and group velocity If we consider a plane wave $u(k; \vec{r}, t)$, the phase velocity is defined as

$$v_{ph} \equiv \frac{\mathrm{d}\omega(k)}{\mathrm{d}k} \tag{B.1}$$

[...]

Appendix C

General formalism of the complete bases of modes

Inner product

In the vector space of the wave functions, which in general are *continuous functions*, we can define the following *inner product*:

$$(\phi, \psi) \underset{def}{=} i \int \phi^* \overleftrightarrow{\partial t} \; \psi \; \mathrm{d}^3\vec{r} \tag{C.1}$$

$$\underset{def}{=} i \int [\phi^*(\partial_t \; \psi) - (\partial_t \phi^*)\psi] \, \mathrm{d}^3\vec{r}. \tag{C.2}$$

Now, let's say that our vector space is the vector space of the pane waves in a continuous (non dispersive) medium (7.2). (comment: in (7.2) the $(2\pi)^3$ comes from the volume the wave propagates in, and the $2\,\omega_{\vec{k}}$ has to do with the relativity invariance).

We want to investigate about the properties of the inner product just defined, applied to these mode functions.

orthonormality

To prove orthonormality let's apply the inner product definition (C.2) to this explicit expression of plane waves:

$$
\begin{aligned}
(u,u') \;=\;& i\int\left[\frac{e^{-i\left(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}\,t\right)}}{\sqrt{(2\pi)^3\;2\,\omega_{\vec{k}}}}\left(-i\omega_{\vec{k}'}\right)\frac{e^{i\left(\vec{k}'\cdot\vec{r}-\omega_{\vec{k}'}\,t\right)}}{\sqrt{(2\pi)^3\;2\,\omega_{\vec{k}'}}}-\right.\\
&\left.-\;\left(i\omega_{\vec{k}}\right)\frac{e^{-i\left(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}\,t\right)}}{\sqrt{(2\pi)^3\;2\,\omega_{\vec{k}}}}\;\frac{e^{i\left(\vec{k}'\cdot\vec{r}-\omega_{\vec{k}'}\,t\right)}}{\sqrt{(2\pi)^3\;2\,\omega_{\vec{k}'}}}\right]\mathrm{d}^3\vec{r}
\end{aligned}
$$

factorizig "$-i$"

$$
\begin{aligned}
(u,u') \;=\;& -ii\int\left[\frac{\omega_{\vec{k}'}\;e^{\left[i\left(\vec{k}'\cdot\vec{r}-\omega_{\vec{k}'}\,t\right)-i\left(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}\,t\right)\right]}}{\sqrt{(2\pi)^3\;2\,\omega_{\vec{k}}}\sqrt{(2\pi)^3\;2\,\omega_{\vec{k}'}}}-\right.\\
&\left.-\;\frac{\left(-\omega_{\vec{k}}\right)\;e^{\left[i\left(\vec{k}'\cdot\vec{r}-\omega_{\vec{k}'}\,t\right)-i\left(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}\,t\right)\right]}}{\sqrt{(2\pi)^3\;2\,\omega_{\vec{k}}}\sqrt{(2\pi)^3\;2\,\omega_{\vec{k}'}}}\right]\mathrm{d}^3\vec{r}\\
=\;& \int\left[\frac{\omega_{\vec{k}'}\;e^{-i\left[\left(\vec{k}-\vec{k}'\right)\cdot\vec{r}-\left(\omega_{\vec{k}}-\omega_{\vec{k}'}\right)t\right]}}{(2\pi)^3\;2\sqrt{\omega_{\vec{k}'}\,\omega_{\vec{k}}}}-\right.\\
&\left.-\;\frac{\left(-\omega_{\vec{k}}\right)\;e^{-i\left[\left(\vec{k}-\vec{k}'\right)\cdot\vec{r}-\left(\omega_{\vec{k}}-\omega_{\vec{k}'}\right)t\right]}}{(2\pi)^3\;2\sqrt{\omega_{\vec{k}'}\,\omega_{\vec{k}}}}\right]\mathrm{d}^3\vec{r}\\
=\;& \int\frac{\omega_{\vec{k}}+\omega_{\vec{k}'}}{(2\pi)^3\;2\sqrt{\omega_{\vec{k}'}\,\omega_{\vec{k}}}}\;e^{-i\left[\left(\vec{k}-\vec{k}'\right)\cdot\vec{r}-\left(\omega_{\vec{k}}-\omega_{\vec{k}'}\right)t\right]}\;\mathrm{d}^3\vec{r}
\end{aligned}
$$

now we use the definition

$$
\int\frac{1}{2\pi}e^{-i(k-k')x}\,\mathrm{d}x\overset{def}{=}\delta(k-k') \tag{C.3}
$$

which holds for each of the three dimensions. So, integrating on $\mathrm{d}^3\vec{r}$

$$
=\delta^3(\vec{k}-\vec{k}')\;\frac{\omega_{\vec{k}}+\omega_{\vec{k}'}}{2\sqrt{\omega_{\vec{k}'}\,\omega_{\vec{k}}}}\;e^{i\left(\omega_{\vec{k}}-\omega_{\vec{k}'}\right)t} \tag{C.4}
$$

where $(2\pi)^3$ in the numerator, that simplifies with the one in the denominator comes from the integration over $\vec{r}$.

Now we are interested in what happens in the limit $\vec{k}\to\vec{k}'$. If we limit ourself at the case of non dispersive media, where the *dispersion relation* $\vec{k}(\omega)$ is linear, as in the vacuum where we have $|\vec{k}|^2=\frac{\omega^2}{c^2}$, we have

$$\lim_{\vec{k}\to\vec{k}'} \frac{\omega_{\vec{k}} + \omega_{\vec{k}'}}{2\sqrt{\omega_{\vec{k}'}\,\omega_{\vec{k}}}} = \frac{2\omega_{\vec{k}'}}{2\omega_{\vec{k}'}} = 1$$
$$\lim_{\vec{k}\to\vec{k}'} e^{i\left(\omega_{\vec{k}}-\omega_{\vec{k}'}\right)t} = 1$$

and so finally we have proved the ortonormality of the plane waves $u(\vec{k};\vec{r},t)$ with respect of the inner product (C.2):

$$(u, u') = \delta^3(\vec{k} - \vec{k}') \tag{C.5}$$

This proof is in the continuous limit. In the discrete limit integrals turn into sums, and the proof is somehow easier, and yields a Kroneker delta.

other properties

$$(\phi, \psi)^* = (\psi, \phi) \tag{C.6}$$
$$(\phi^*, \psi^*) = -(\phi, \psi) \tag{C.7}$$
$$(u, u^*) = 0 \tag{C.8}$$

where we remember that the generic plane wave depends on the wave vector, as well as the space and time: $u = u(\vec{k};\vec{r},t)$.

C.0.1 Superposition of plane waves

Let's build a (continuous) superposition of plane waves (we can use also the name of *mode functions* instead of plane waves) $u(\vec{k};\vec{r},t)$ in the following way:

$$f(\vec{r},t) = \int \left[\alpha(\vec{k})\; u(\vec{k};\vec{r},t) \;+\; \beta(\vec{k})\; u^*(\vec{k};\vec{r},t)\right] \mathrm{d}\vec{k}. \tag{C.9}$$

We want now to find the relationship between the coefficients $\alpha(\vec{k})$ and $\beta(\vec{k})$ and the function $f(\vec{r},t)$. We can proof that:

$$\begin{cases} \alpha(\vec{l}) = (u(\vec{l}), f) \\ \beta(\vec{l}) = -(u^*(\vec{l}), f) \end{cases} \tag{C.10}$$

Let's see:

$$
\begin{aligned}
(u(\vec{l}), f) &= \left(u(\vec{l})\ ,\ \int \left[\alpha(\vec{k})\, u(\vec{k}) \ +\ \beta(\vec{k})\, u^*(\vec{k})\right] \mathrm{d}^3\vec{k}\right) && \text{(C.11)}\\
&= \int \left[\alpha(\vec{k}) \underbrace{\left(u(\vec{l}), u(\vec{k})\right)}_{(a)} + \beta(\vec{k}) \underbrace{\left(u(\vec{l}), u^*(\vec{k})\right)}_{(b)}\right] \mathrm{d}^3\vec{k} && \text{(C.12)}\\
&= \int \alpha(\vec{k})\ \delta(\vec{k}-\vec{k}')\mathrm{d}^3\vec{k} && \text{(C.13)}\\
&= \alpha(\vec{l}) && \text{(C.14)}
\end{aligned}
$$

where we have used the orthonormality (C.5) so that $(a) = \delta(\vec{k}-\vec{k}')$ and the property (C.8) so that $(b) = 0$.

completeness

Let's now consider *two* mode functions f and g, that are two different superposition of plane waves u, as in (C.9), and let's write their inner product:

$$
(g, f) \tag{C.15}
$$

let's rewrite the expansion of f (C.9) using the relations (C.10)

$$
f(\vec{r}, t) = \int \left[\left(u(\vec{k}), f\right)\ u(\vec{k})\ \ -\ \ \left(u^*(\vec{k}), f\right)\ u^*(\vec{k})\right] \mathrm{d}^3\vec{k} \tag{C.16}
$$

and use it in the inner product

$$
\begin{aligned}
(g, f) &= \int \left[\left(u(\vec{k}), f\right)\ \left(g, u(\vec{k})\right)\ -\ \left(u^*(\vec{k}), f\right)\ \left(g, u^*(\vec{k})\right)\right] \mathrm{d}^3\vec{k} && \text{(C.17)}\\
&= \int \left[\left(g, u(\vec{k})\right)\ \left(u(\vec{k}), f\right)\ -\ \left(g, u^*(\vec{k})\right)\ \left(u^*(\vec{k}), f\right)\right] \mathrm{d}^3\vec{k}. && \text{(C.18)}
\end{aligned}
$$

Summarizing, we can rewrite the last result

$$(g,f) = \Big(g \Big| \underbrace{\int \Big[\Big| u(\vec{k})\Big) \Big(u(\vec{k}) \Big| - \Big| u^*(\vec{k})\Big) \Big(u^*(\vec{k})\Big| \Big] \mathrm{d}^3\vec{k}}_{\text{identity}} \Big| f\Big) \tag{C.19}$$

This is a bit "handwaving", but is a way to express the *completeness* of the basis of plane waves.

completeness different ways to express the concept of completeness:

- is possible to "plug" the identity that appears in (C.19) anywhere in products
- - if there where functions that are not in the for of a superposition as in (C.9), they are not part of the space of the mode functions
  - AND
  - any function f of the form (C.9) will be in the space, and will be completely determined by the coefficients $\alpha(\vec{k}) = (u(\vec{k}), f)$ and $\beta(\vec{k}) = -(u^*(\vec{k}), f)$

So *plane waves are a complete orthonormal basis*, and can be used to build wave packets.

C.0.2 Mode functions transformations

Here we want to develop some formal tools to transform the *complete set of modes* of the plane waves, into a new complete set of modes.

Let's consider two *unitary operators* on the space of the mode functions, V and W, that produce a "basis transformation" as follows:

$$v(\vec{j};\vec{r},t) = \int \Big[V\left(\vec{j},\vec{k}\right) u\left(\vec{k};\vec{r},t\right) + W^*\left(\vec{j},\vec{k}\right) u^*\left(\vec{k};\vec{r},t\right)\Big] \mathrm{d}^3\vec{k} \tag{C.20}$$

Since V and W are unitary, the set of $\left\{v(\vec{j};\vec{r},t), v^*(\vec{j};\vec{r},t)\right\}_{\vec{j}}$ is a new *complete orthonormal basis* of the mode functions space.

To give an intuition, this can be seen as the continuous version of the following discrete relation:

$$v_i = \sum_k V_{j,k}\; u_k \tag{C.21}$$

The convention used is to express continuous variables as arguments, and the discrete indices as subscripts. But although we are dealing with continuous variables, we can imagine $V(\vec{j},\vec{k})$ and $W(\vec{j},\vec{k})$ as "infinite matrices" (since we have two "labels" running). Since the only requirement in order for $\left\{v(\vec{j};\vec{r},t), v^*(\vec{j};\vec{r},t)\right\}_{\vec{j}}$ to be a complete orthonormal basis is for $V(\vec{j},\vec{k})$ and $W(\vec{j},\vec{k})$ to be unitary, "fiddling" with the matrix elements of these operators we have a very large choice for new basis. To give a concrete example we could transform the basis and go from the plane waves (7.2) to the Hermite polinomials.

C.0.3 Creation and annihilation operators

Up to now, all we have said is strictly "classical". We can apply the content of this section so far to *classical wave optics*. Now we want apply what we found so far to the quantized description of electromagnetic field. So, as described in subsection 6.3, we start from a superposition of mode functions, that we call $A(\vec{r},t)$ (something like (C.9)):

$$A(\vec{r},t) = \int \left[a(\vec{k})\, u(\vec{k};\vec{r},t) \;+\; a^{+}(\vec{k})\, u^*(\vec{k};\vec{r},t)\right] \mathrm{d}^3\vec{k} \tag{C.22}$$

and then we *promote* the coefficients a and a^+ to *operators*:

$$\hat{A}(\vec{r},t) = \int \left[\hat{a}(\vec{k})\, u(\vec{k};\vec{r},t) \;+\; \hat{a}^{\dagger}(\vec{k})\, u^*(\vec{k};\vec{r},t)\right] \mathrm{d}^3\vec{k} \tag{C.23}$$

(comments:

- the second part of the integral has the purpose to make $\hat{A}$ an hermitian operator

- here we are neglecting the polarization, or we are incorporating it in the indices

).

Now, let's imagine that we perform a basis change $u \to v$ as in (C.20), and rewrite $\hat{A}$ in the new basis:

$$\hat{A}(\vec{r},t) = \int \left[\hat{b}(\vec{j})\, v(\vec{j};\vec{r},t) \;+\; \hat{b}^{\dagger}(\vec{j})\, v^{*}(\vec{j};\vec{r},t)\right] \mathrm{d}^{3}\vec{j} \tag{C.24}$$

where the basis has changed, and so have the creation and annihilation operators, that play the role of "coefficients".

Now, in analogy with (C.10), we can write an expression of the creation and annihilation operators in terms of the basis vectors and the field operator, and we can use first the basis $\left\{u(\vec{l}), u^{*}(\vec{l})\right\}_{\vec{l}}$:

$$\begin{cases} \hat{a}(\vec{l}) = \left(u(\vec{l}), \hat{A}\right) \\ \hat{a}^{\dagger}(\vec{l}) = -\left(u^{*}(\vec{l}), \hat{A}\right) \end{cases} \tag{C.25}$$

and then the basis $\left\{v(\vec{j}), v^{*}(\vec{j})\right\}_{\vec{j}}$:

$$\begin{cases} \hat{b}(\vec{j}) = \left(v(\vec{j}), \hat{A}\right) \\ \hat{b}^{\dagger}(\vec{j}) = -\left(v^{*}(\vec{j}), \hat{A}\right). \end{cases} \tag{C.26}$$

Finally, to have a transformation rule that shows us how to express the creation/annihilation operators related to the first basis, to the creation/annihilation operators related to the second basis, we can "plug" the expression of $\hat{A}$ as superposition of $\left\{u(\vec{l}), u^{*}(\vec{l})\right\}_{\vec{l}}$, into the expression of the creation/annihilation operators related to $\left\{v(\vec{j}), v^{*}(\vec{j})\right\}_{\vec{j}}$:

$$\begin{aligned}
\hat{b}(\vec{j}) &= (v(\vec{j}), \hat{A}) \\
&= \left(v(\vec{j}), \int \left[\hat{a}(\vec{k})\, u(\vec{k}) \;+\; \hat{a}^{\dagger}(\vec{k})\, u^{*}(\vec{k})\right] \mathrm{d}^{3}\vec{k}\right) \\
&= \int \left[\hat{a}(\vec{k}) \left(v(\vec{j}), u(\vec{k})\right) \;+\; \hat{a}^{\dagger}(\vec{k}) \left(v(\vec{j}), u^{*}(\vec{k})\right)\right] \mathrm{d}^{3}\vec{k} \\
&= \int \left[V^{-1}(\vec{j},\vec{k})\, \hat{a}(\vec{k}) \;+\; W^{-1}(\vec{j},\vec{k})\, \hat{a}^{\dagger}(\vec{k})\right] \mathrm{d}^{3}\vec{k}
\end{aligned}$$

and similarly

$$\begin{aligned}\hat{b}^\dagger(\vec{j}) &= -(v^*(\vec{j}), \hat{A}) \\ &= -\int \left[\hat{a}(\vec{k}) \left(v^*(\vec{j}), u(\vec{k})\right) + \hat{a}^\dagger(\vec{k}) \left(v^*(\vec{j}), u^*(\vec{k})\right)\right] \mathrm{d}^3\vec{k}\end{aligned}$$

here we apply the property (C.7) of the inner product

$$\begin{aligned}&= \int \left[\hat{a}(\vec{k}) \left(v(\vec{j}), u^*(\vec{k})\right) + \hat{a}^\dagger(\vec{k}) \left(v(\vec{j}), u(\vec{k})\right)\right] \mathrm{d}^3\vec{k} \\ &= \int \left[W^{-1}(\vec{j}, \vec{k})\, \hat{a}(\vec{k}) + V^{-1}(\vec{j}, \vec{k})\, \hat{a}^\dagger(\vec{k})\right] \mathrm{d}^3\vec{k}\end{aligned}$$

So we can see that

V "mixes" $\hat{a} \leftrightarrow \hat{b}$ and $\hat{a}^\dagger \leftrightarrow \hat{b}^\dagger$

W "mixes" $\hat{a} \leftrightarrow \hat{b}^\dagger$ and $\hat{a}^\dagger \leftrightarrow \hat{b}$.

In a discrete formalism we would have:

$$\begin{cases} \hat{b}(\vec{j}) = \sum_{\vec{k}} \left[V^{-1}_{\vec{j}\,\vec{k}}\, \hat{a}(\vec{k})\ W^{-1}_{\vec{j}\,\vec{k}}\, \hat{a}^\dagger(\vec{k})\right] \\ \hat{b}^\dagger(\vec{j}) = \sum_{\vec{k}} \left[W^{-1}_{\vec{j}\,\vec{k}}\, \hat{a}(\vec{k})\ V^{-1}_{\vec{j}\,\vec{k}}\, \hat{a}^\dagger(\vec{k})\right]. \end{cases} \tag{C.27}$$

These formulas give the formal relations between the creation/annihilation operators, but is worth mentioning that in order for the "new" operators to be proper creation/annihilation operators, we have to require explicitly that they satisfy the *commutation relations* (cfr (6.14)):

$$\begin{cases} \left[\hat{b}(\vec{j}), \hat{b}(\vec{j}')\right] = \left[\hat{b}^\dagger(\vec{j}), \hat{b}^\dagger(\vec{j}')\right] = 0 \\ \left[\hat{b}^\dagger(\vec{j}), \hat{b}(\vec{j}')\right] = \left[\hat{b}(\vec{j}), \hat{b}^\dagger(\vec{j}')\right] = \delta(\vec{j} - \vec{j}') \end{cases} \tag{C.28}$$

(check: here I am not sure, because maybe the commutation relations can be derived from the unitarity of V and W)

Appendix D

Bloch sphere

see notes from:

2011-10-03 rQKD-16, t=23', page 2

2011-11-21 TbPBB84-04, t= 21' 50", IMG_0717.jpg

If we want to represent a qubit, i.e. a quantum system with an Hilbert space of dimension 2, we are going to use the following *computational basis*

$$\{|0\rangle, |1\rangle\} \tag{D.1}$$

The generic state is a linear combination of the two states $|0\rangle$and$|1\rangle$ of the computational basis:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{D.2}$$

At first, let's set $\varphi = 0$, i.e. let's consider only states along a "meridian" of the spere. In this case the normalization condition is

$$|\alpha|^2+|\beta|^2= 1 \tag{D.3}$$

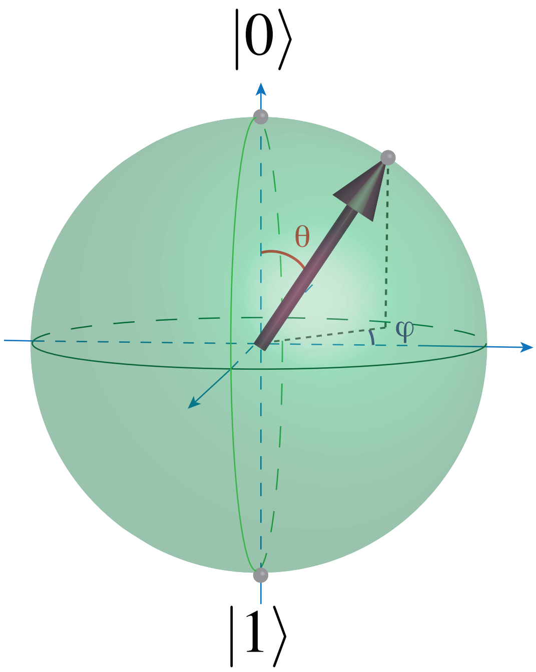

Figure D.1: bloch sphere

which, considering the trigonometric relation $\cos^2 * + \sin^2 * = 1$ can be written as

$$\begin{aligned} |\psi\rangle &= \cos *|0\rangle + \sin *|1\rangle \\ \alpha &= \cos * \\ \beta &= \sin * \end{aligned} \quad \text{(D.4)}$$

where $*$ is a suitable angle. Since we want to be $\theta \in [0, \pi]$ (see figure D.1) and

$$\begin{cases} \theta = 0 \ \Rightarrow \ (\alpha = 1; \beta = 0) \\ \theta = \pi \ \Rightarrow \ (\alpha = 0; \beta = 1) \end{cases} \quad \text{(D.5)}$$

we choose $* = \frac{\theta}{2}$ which fulfills the requirements:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle. \quad \text{(D.6)}$$

D.1 Visualizing the probabilities

The probabilities of measuring $|0\rangle$ and $|1\rangle$ can be "visualized" easily considering the projection on the vertical axis of the Bloch sphere, of the generic vector's apex (see figure D.2): the square of the distance from this point and $|1\rangle$ on the axis will be *proportional to* the probability of measuring $|0\rangle$, and similarly for $|0\rangle$:

$$\begin{aligned} p(0|\psi) &= 1 - \cos\theta = 2\sin^2\tfrac{\theta}{2} \\ p(1|\psi) &= 1 + \cos\theta = 2\cos^2\tfrac{\theta}{2} \end{aligned} \tag{D.7}$$

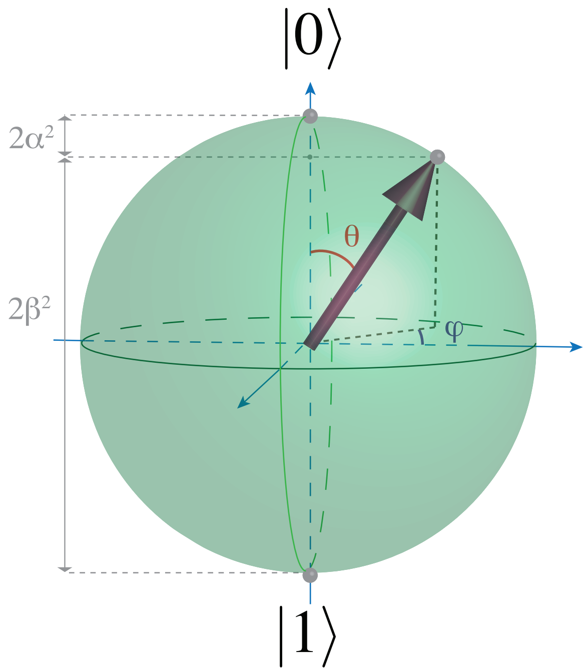

Figure D.2: probabilities of a generic qubit state seen as (proportional to the) projections on the axis

To have the probabilities a renormalization is needed (total probability = 1)

Appendix E

Tensor product and direct sum

all the chapter but the last section is from discussions with Pieter Kok (6-11 June 2007)

E.1 Tensor product

If you have a "big system", made of two or more subsystems, the hilbert space of the big system can be written as *tensor product* of smaller Hilbert spaces:

$$\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c \otimes \dots \tag{E.1}$$

For example, if I have two systems, each of which live in an Hilbert space of dimension 2, i.e with a basis of two state vectors:

$$\mathcal{H}_a = \{|0\rangle_a, |1\rangle_a\}\mathcal{H}_b = \{|0\rangle_b, |1\rangle_b\} \tag{E.2}$$

then the Hilbert space describing both systems (total system) is the tensor product $\mathcal{H}_a \otimes \mathcal{H}_b$, and the basis of this total Hilbert space is the tensor product of the single bases. Since it is a product, the " cross terms" will appear:

$$\begin{aligned} &\{|0\rangle_a, |1\rangle_a\} \otimes \{|0\rangle_b, |1\rangle_b\} \\ =&\{|0\rangle_a \otimes |0\rangle_b, ...\} \\ =&\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}. \end{aligned} \tag{E.3}$$

The meaning of the cross terms can be understood intuitively with the necessity to represent all the possible states, where each subsystem is in each possible state. Also the states where "a" is in $|0\rangle_a$ and "b" is in $|1\rangle_b$ and vice versa, have to be represented.

E.1.1 Spaces and dimensions for tensor product

The dimension of the (space obtained as) tensor product is the *product* of the dimensions of the "factors": if $\mathcal{H}_a$ has dimension n and $\mathcal{H}_b$ has dimension m, the dimension d of $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ is $d = n \times m$.

To have an intuition about this, we can think at the fact that given the two bases of $\mathcal{H}_a$ and $\mathcal{H}_b$, the basis of $\mathcal{H}$ is obtained considering all the possible couples of elements one for each basis. This happens to be the only way to build the vector space obtained with the concatenation of the vectors of the two initial spaces, and preserve *linearity* in doing so. (for this last statement, see the class of Patrick Hayden, (Montréal 2013) lecture 09)

E.2 Direct sum

In general a matrix acts on basis elements, a turn them in other elements. So we can put a "label" on the left, indicating "on which element of the basis that row acts".

$$\left(\begin{array}{cc|cc} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ \hline m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{array}\right) \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \tag{E.4}$$

A generic vector will be the sum of each basis elements.

If we have a *block diagonal matrix*, we see that each block acts on only a subspace. The subspaces are never mixed. This means that, for what matters to this matrix, there are two separate parts of the Hilbert space. We can separate those two parts, and to "reconstruct" the whole space, we have to do a *direct sum*

$$\left(\begin{array}{cc|cc} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ \hline 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{array}\right) \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} = M_{12} \oplus M_{34} \tag{E.5}$$

E.2.1 Spaces and dimensions for direct sum

If we introduce an Hilbert space which is the direct sum of two other Hilbert spaces:

$$\mathcal{H} = \mathcal{H}_\alpha \oplus \mathcal{H}_\beta$$

this means that each element of $\mathcal{H}$ can be written as a linear combination of two elements, one in $\mathcal{H}_\alpha$ and one in $\mathcal{H}_\beta$:

$$\forall\ |\psi_\alpha\rangle \in \mathcal{H}_\alpha, |\psi_\beta\rangle \in \mathcal{H}_\beta; \qquad \mathcal{H} = \text{span}\{\alpha|\psi_\alpha\rangle + \beta|\psi_\beta\rangle\}_{\alpha,\beta\in\mathbb{C}} \tag{E.6}$$

This is the same (at finite dimensions) as the two dimensional plane being spanned by two single dimension subspaces:

$$\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R} \tag{E.7}$$

The dimension of the space $\mathcal{H} = \mathcal{H}_\alpha \oplus \mathcal{H}_\beta$ is the *sum* of the dimension of the spaces $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$. (to be checked!).

To have an intuition about this, we can think at the fact that given the two bases of $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$, the basis of $\mathcal{H}$ is obtained as the two bases one after the other, *i.e.* the union of the two sets.

E.2.2 physical intuition about the direct sum

the following notion is from a discussion with Frédéric Grosshans (29 Nov 2011 - at lunch)

The direct sum is to "combine" two "things" in such a way that they interfere, i.e. they are summed as amplitudes, and not as intensities, conserving the complex phases. In other words, the direct sum conveys the idea of coherence.

E.3 Final comparisons between T.P. and D.S.

the following notion is from discussions (with Abdulrahman Al-lahham and Cosmo Lupo, May 2013) around the QMem access problem (May 2013)

Another way to see this, is to compare it with the tensor product: if we consider the tensor product of several "components", and if the original components have each a different (complex) coefficient, in the tensor product the coefficients are factorized, and we end up with a single coefficient:

$$a_1 |x_1\rangle \otimes \ldots \otimes a_n |x_n\rangle = (a_1 \cdot \ldots \cdot a_n) |x_1 \ldots x_n\rangle \tag{E.8}$$

So the information about the coefficients (and the phases) are lost). In this sense, the direct sum "preserves" the info about the phases, and allows interference, while the tensor product doesn't.

It is worth noting that $|i, j\rangle \neq |i, 0\rangle + |0, j\rangle$, so $\mathcal{F}^{ab} = \mathcal{E}^{ab}_{0\tilde{1}} \oplus \mathcal{E}^{ab}_{\tilde{1}0}$

Appendix F

Fock spaces

A Foch space is a bigger structure, with respect to Hilbert spaces, which is used tu represent systems with a variable number of components (particles).

The typical system with such characteristic is the e.m. field.

F.1 n-particles Hilbert space

$$\mathcal{H}_n = \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}}_{\text{n times}} \tag{F.1}$$

an element of this space is a state with a known number of particles (photons), in a known mode.

F.2 single mode Fock space

Several "n-particles hilbert spaces" (previous sections) form a Fock space, where the number of particles is undetermined (creation and annihilation):

$$\begin{aligned} \mathcal{F}_a &= \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_n \oplus \ldots \\ &= \bigoplus_{n=0}^{\infty} H_n \end{aligned} \tag{F.2}$$

F.3 multi-mode Fock space

$$
\begin{aligned}
\mathcal{F}^m & \\
= & \\
\mathcal{F}_a &= \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_n \oplus \ldots \\
\otimes & \\
\mathcal{F}_b &= \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_n \oplus \ldots \\
\otimes & \\
\mathcal{F}_c &= \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_n \oplus \ldots \\
\otimes & \\
\vdots &
\end{aligned}
$$

in more compact notation:

$$
\begin{aligned}
\mathcal{F}^m &= \bigotimes_i \mathcal{F}_i \qquad \text{(tensor prod. of single mode Fock sp.)} \\
&= \bigotimes_i \left[\overbrace{\mathcal{H}^{\otimes 0} \oplus \mathcal{H}^{\otimes 1} \oplus \mathcal{H}^{\otimes 2} \oplus \ldots \oplus \mathcal{H}^{\otimes n} \oplus \ldots}^{\mathcal{F}_i} \right]_i \\
&= \bigotimes_i \left[\overbrace{\bigoplus_n \mathcal{H}^{\otimes n}}^{\mathcal{F}_i} \right]_i \\
&= \bigotimes_i \left[\bigoplus_n \overbrace{\left(\bigotimes_{j=0}^{n} \mathcal{H}_j \right)}^{\mathcal{H}^{\otimes n}} \right]_i
\end{aligned}
$$

Appendix G

Time-bin appendices

G.1 Action of creation and annihilation operators

Once the *multimode Fock space* has been defined (6.17), is possible to give a precise definition of the action of the creation and annihilation operators. A single mode creation (annihilation) operator $\hat{a}_m^\dagger$ ($\hat{a}_m$) is defined on a single mode Fock space (6.16), but we have seen that an actual state of the e.m. field is an element of the bigger multimode Fock space. So to make everything correct, we have to define an extension of $\hat{a}_m^\dagger$ and $\hat{a}_m$ as follow

$$\hat{a}_m^{ext} \equiv \mathbb{I}_0 \otimes \mathbb{I}_1 \otimes \ldots \otimes \hat{a}_m \otimes \ldots \otimes \mathbb{I}_n \ldots \tag{G.1}$$

When the annihilation operator related to the mode n is applied to a state with a single photon in the mode $m \neq n$ the usual notation is $\hat{a}_n|1_m\rangle$. However this notation is too synthetic.

A more detailed notation is obtained if we write the state of the system as an element of the multimode Fock space, writing explicitly all the other modes in the ground state, and we use the extended creation operator. We see that the state with one photon in mode m will remain unchanged by the relative identity operator, but the annihilation operator in mode n will act on the relative *vacuum state*, producing a "scalar zero"; then the tensor product of this scalar zero with the other states will produce a scalar zero:

$$
\begin{aligned}
\hat{a}_n|1\rangle_{m\neq n} &= [\mathbb{I}_0 \otimes \mathbb{I}_1 \otimes \ldots \otimes \hat{a}_n \otimes \ldots \otimes \mathbb{I}_m \ldots][|0\rangle_0 \otimes |0\rangle_1 \otimes \ldots \otimes |1\rangle_m \ldots \otimes |0\rangle_n \ \otimes \ldots] \\
&= \mathbb{I}_0|0\rangle_0 \otimes \mathbb{I}_1|0\rangle_1 \ldots \otimes \mathbb{I}_m|1\rangle_m \ldots \otimes \hat{a}_n|0\rangle_n \ \otimes \ldots |0\rangle_{n+1} \ldots \\
&= |0\rangle_0 \otimes |0\rangle_1 \ldots \otimes |1\rangle_m \ldots \otimes \ 0 \ \otimes \ldots \\
&= 0.
\end{aligned}
$$

On the other hand it is trivial to see that the action of the annihilation operator related to the mode n, on a state with a single photon in the same mode n, we obtain the vacuum state of the whole multimode Fock space:

$$
\begin{aligned}
\hat{a}_n|1\rangle_n &= \hat{a}_n[|0\rangle_0 \otimes |0\rangle_1 \ldots \otimes |1\rangle_n \ \otimes |0\rangle_{n+1} \ldots] \\
&= \hat{a}_n|0\rangle_0 \otimes \hat{a}_n|0\rangle_1 \ldots \otimes \hat{a}_n|1\rangle_n \ \otimes \hat{a}_n|0\rangle_{n+1} \ldots \\
&= |0\rangle_0 \otimes |0\rangle_1 \ldots \otimes 1|0\rangle_n \ \otimes |0\rangle_{n+1} \ldots \\
&= |0\rangle_0 \otimes |0\rangle_1 \ldots \otimes |0\rangle_n \ \otimes |0\rangle_{n+1} \ldots \\
&= |0\rangle
\end{aligned}
$$

We can summarize those results in the following compact form using the Kronecker delta:

$$
\hat{a}_n|1\rangle_m = \delta_{nm}|0\rangle \tag{G.2}
$$

G.2 Appendix on detailed calculations

G.2.1 Expectation value of the number operator on a localized single photon state

$n(t) = \langle 1; f|\hat{n}|1; f\rangle.$

First we apply the definition (7.12) for the localized single photon state

$$\begin{aligned} n(t) &= \left[\sum_{n=1}^{\infty}\langle 0|\hat{a}_{\omega_n} f_n^*\right] \hat{n}(t) \left[\sum_{m=1}^{\infty} f_m \hat{a}^{\dagger}_{\omega_m}|0\rangle\right] \\ &= \sum_{n=1}^{\infty} f_n^* f_m \langle 0|\hat{a}_{\omega_n} \overbrace{\hat{a}^{\dagger}(t)\hat{a}(t)}\, \hat{a}^{\dagger}_{\omega_m}|0\rangle \end{aligned}$$

now we apply the expansion (7.3)

$$\begin{aligned} n(t) &= \sum_{n,m,k,l=1}^{\infty} f_n^* f_m \langle 0|\hat{a}_{\omega_n} \underbrace{\hat{a}^{\dagger}_{\omega_k} e^{i\,\omega_k t}}\; \underbrace{\hat{a}_{\omega_l} e^{-i\,\omega_l t}}\, \hat{a}^{\dagger}_{\omega_m}|0\rangle \\ &= \sum_{n,m,k,l=1}^{\infty} f_n^* f_m\, e^{i\,\omega_k t}\, e^{-i\,\omega_l t} \langle 0|\hat{a}_{\omega_n} \hat{a}^{\dagger}_{\omega_k} \hat{a}_{\omega_l} \hat{a}^{\dagger}_{\omega_m}|0\rangle \\ &= \sum_{n,m,k,l=1}^{\infty} f_n^* f_m\, e^{i\,\omega_k t}\, e^{-i\,\omega_l t}\; {}_n\langle 1|\hat{a}^{\dagger}_{\omega_k} \hat{a}_{\omega_l}|1\rangle_m \end{aligned}$$

now we have to apply the annihilation operators related to a single mode on a state with a single photon in a single mode, and as shown in (G.2) the result is a delta times the vacuum state $|0\rangle = |0\rangle_0 \otimes |0\rangle_1 \otimes \ldots \otimes |0\rangle_n \ldots$

$$\begin{aligned} &= \sum_{n,m,k,l=1}^{\infty} f_n^* f_m\, e^{i\,\omega_k t}\, e^{-i\,\omega_l t} \langle 0|\delta_{kn}\delta_{lm}|0\rangle \\ &= \sum_{n,m,k,l=1}^{\infty} f_n^* f_m\, e^{i\,\omega_k t}\, e^{-i\,\omega_l t} \delta_{kn}\delta_{lm}\langle 0|0\rangle \end{aligned}$$

the vacuum state is normalized, so $\langle 0|0\rangle = 1$

$$= \sum_{n,m,k,l=1}^{\infty} f_n^* f_m \, e^{i\,\omega_k t} \, e^{-i\,\omega_l t} \delta_{kn} \delta_{lm}$$
$$= \sum_{n,m=1}^{\infty} f_n^* f_m \, e^{i\,\omega_n t} \, e^{-i\,\omega_m t}$$
$$= \left(\sum_{n=1}^{\infty} f_n^* \, e^{i\,\omega_n t} \right) \left(\sum_{m=1}^{\infty} f_m \, e^{-i\,\omega_m t} \right)$$

now, since the indices of the sums are dummy indices, and the arguments of the sums are one the complex conjugate of the other, we finally have

$$n(t) = \langle 1; f | \hat{n} | 1; f \rangle = \sum_{n=1}^{\infty} |f_n \, e^{-i\,\omega_n t}|^2. \tag{G.3}$$

Comments

We have computed the expectation value of the number operator over the state (7.1) (localized single photon). This function gives the most probable result if we measure the number of photons of the system when it is in the "localized single photon" state, as a function of time. We can also think at this quantity as the probability to count a photon, as a function o time.

This function is periodic. This comes from the fact that in the "anti-transform" (7.4) we have discretized the time, and we have used a sum instead of an integral.

summary It is worth to summarize and stress what is the rationale behind this calculation.

We have two different *expansions*:

- we *expand the state* of the radiation field over the plane waves (7.1) : $|1; f\rangle = \sum_{n=1}^{\infty} f_n \hat{a}^\dagger_{\omega_n} |0\rangle$. This is to have a *spatially localized* state (wave packet).
- we *expand the operator* that measures the number of photons over the plane waves (7.3): $\hat{a}(t) = \sum_{n=1}^{\infty} \hat{a}_n e^{-i\,\omega_n t}$. This is indeed the expansion of the annihilation,

because we can express the "number operator" only in terms of the creation and annihilation operators. This second expansion is the expansion of the time dependent annihilation operator over the single mode annihilation operators; we do this because we only know the action of these single mode operators on the single mode states.

The first expansion, the expansion of the state, has coefficients f_m.

The second expansion, the expansion of the annihilation (creation) operators has (time dependent) coefficients $e^{\pm i\,\omega_n t}$, and so it can be seen as a Fourier transform. This form of the coefficients comes from the choice of the plane waves as Slater modes.

explicit expression of the coefficients Some explicit expressions of packet coefficients are:

$$f_m^{N_{max}} = \frac{1}{\sqrt{1-(1-\mu)^{N_{max}}}} \begin{pmatrix} N_{max} \\ m \end{pmatrix}^{\frac{1}{2}} \mu^{\frac{m}{2}} (1-\mu)^{\frac{N_{max}-m}{2}} \tag{G.4}$$

and

$$f_n^{N_{max}} = \frac{1}{A} \frac{\sqrt{\mu}}{\mu + i\,n} \tag{G.5}$$

where, in the limit $N_{max} \to \infty$,

$$A = \frac{\pi e^{\mu\pi}}{2\sinh(\mu\pi)} - \frac{1}{2\mu} \tag{G.6}$$

G.2.2 a(t) as function of time-bins operators

We can obtain a new expression of $\hat{a}(t)$ in terms of the $\hat{b}_\mu$ operators, substituting (7.6) into (7.3). For simplicity we substitue $\omega_n = n$:

$$
\begin{aligned}
\hat{a}(t) &= \sum_{n=1}^{\infty} \hat{a}_n e^{-i\,n\,t} \\
&= \sum_{n=1}^{\infty} \overbrace{\frac{1}{\sqrt{N}} \sum_{\mu=1}^{N} e^{i\,\tau\,\mu\,n}\; \hat{b}_\mu}\; e^{-i\,n\,t} \\
&= \frac{1}{\sqrt{N}} \sum_{n=1}^{\infty} \sum_{\mu=1}^{N} e^{i\,\tau\,\mu\,n}\, e^{-i\,n\,t}\; \hat{b}_\mu \\
&= \frac{1}{\sqrt{N}} \sum_{\mu=1}^{N} \left[\sum_{n=1}^{\infty} \left[e^{i\,(\tau\mu - t)} \right]^n \right]\; \hat{b}_\mu
\end{aligned}
$$

here we recognize the geometric series $\sum_{n=1}^{\infty} \lambda^n = \frac{\lambda}{1-\lambda}$ so we can substitute the content of the outer brackets

$$
\begin{aligned}
\sum_{n=1}^{\infty} \left[e^{i\,(\tau\mu - t)} \right]^n &= \frac{e^{i\,(\tau\mu - t)}}{1 - e^{i\,(\tau\mu - t)}} \\
&= \frac{e^{i\,(\tau\mu - t)}}{1 - e^{i\,(\tau\mu - t)}}
\end{aligned}
$$

so that

$$
\hat{a}(t) = \frac{1}{\sqrt{N}} \sum_{\mu=1}^{N} \frac{e^{i\,(\tau\mu - t)}}{1 - e^{i\,(\tau\mu - t)}}\; \hat{b}_\mu
$$

and if we define

$$
g_\mu(t) \equiv \frac{1}{\sqrt{N}} \frac{e^{i\,(\tau\mu - t)}}{1 - e^{i\,(\tau\mu - t)}} \tag{G.7}
$$

with $\tau \equiv \frac{2\pi}{N}$ being the *time-bin width*, we finally have

$$
\hat{a}(t) = \sum_{\mu=1}^{N} g_\mu(t)\, \hat{b}_\mu. \tag{G.8}
$$

G.2.3 Commutation relations of the time-bins operators

Let's compute the commutation relations of the time-bin operators:

$$\begin{aligned}\left[\hat{b}_\mu, \hat{b}_\nu\right] &= \left[\left(\frac{1}{\sqrt{N}}\sum_{m=1}^{N} e^{-i\,\tau\mu\, m}\ \hat{a}_m\right), \left(\frac{1}{\sqrt{N}}\sum_{k=1}^{N} e^{-i\,\tau\nu\, k}\ \hat{a}_k\right)\right] \\ &= \frac{1}{N}\, e^{i\,\tau(\mu\, m-\nu k)} \sum_{m,k=1}^{N} \left[\hat{a}_m, \hat{a}_k\right]\end{aligned}$$

here we can use what we have seen in section G.1: since $\hat{a}_m$ and $\hat{a}_k$ with $m \neq k$ are defined on different subspaces of the multimode Fock space (6.17), they commute ($[\hat{a}_m, \hat{a}_k] = 0 \quad \forall m \neq k$), while for $m = k$ we know that an operator commutes with itself. So this commutator is zero.

Similar calculations lead to the same result for the creation operators.

For the mixed commutator we have:

$$\begin{aligned}\left[\hat{b}_\mu, \hat{b}_\nu^\dagger\right] &= \left[\left(\frac{1}{\sqrt{N}}\sum_{m=1}^{N} e^{-i\,\tau\mu\, m}\ \hat{a}_m\right), \left(\frac{1}{\sqrt{N}}\sum_{k=1}^{N} e^{i\,\tau\nu\, k}\ \hat{a}_k^\dagger\right)\right] \\ &= \frac{1}{N}\sum_{m,k=1}^{N} e^{i\,\tau\,(\nu k-\mu m)} \left[\hat{a}_m, \hat{a}_k^\dagger\right] \\ &= \frac{1}{N}\sum_{m,k=1}^{N} e^{i\,\tau\,(\nu k-\mu m)} \delta_{m,k} \\ &= \frac{1}{N}\sum_{m=1}^{N} e^{i\,\tau\, m\,(\nu-\mu)} \\ &= \delta_{\nu,\mu}.\end{aligned}$$

Summarizing we can conclude that the time-bin operators have the same commutation relations (the same algebra) of the single mode creation and annihilation operators:

$$\begin{cases}\left[\hat{b}_\nu, \hat{b}_\mu\right] = \left[\hat{b}_\nu^\dagger, \hat{b}_\mu^\dagger\right] = 0 \\ \left[\hat{b}_\nu, \hat{b}_\mu^\dagger\right] = \left[\hat{b}_\nu^\dagger, \hat{b}_\mu\right] = \delta_{\nu,\mu}.\end{cases} \tag{G.9}$$

G.2.4 time-bin annihilation operator acting on single photons

We want to compute the expectation value of the number operator on a "time-bin" single photon state. But before we start the calculation, is worth pointing out that, similarly to (G.2), we have

$$\hat{b}_\nu|\tilde{1}\rangle_\mu = \delta_{\nu\mu}|0\rangle. \tag{G.10}$$

Explicitly:

$$\begin{aligned}
\hat{b}_\nu|\tilde{1}\rangle_\mu &= \overbrace{\frac{1}{\sqrt{N}}\sum_m e^{-i\tau\nu m}\hat{a}_m}\overbrace{\frac{1}{\sqrt{N}}\sum_k e^{i\tau\mu k}\hat{a}_k^\dagger|0\rangle} \\
&= \frac{1}{\sqrt{N}}\sum_m e^{-i\tau\nu m}\hat{a}_m\frac{1}{\sqrt{N}}\sum_k e^{i\tau\mu k}|\tilde{1}_k\rangle \\
&= \frac{1}{N}\sum_{m,k} e^{i\tau(\mu k-\nu m)}\hat{a}_m|\tilde{1}_k\rangle \\
&= \frac{1}{N}\sum_{m,k} e^{i\tau(\mu k-\nu m)}\delta_{m,k}|0\rangle \\
&= \frac{1}{N}\sum_{m} e^{i\tau(\mu m-\nu m)}|0\rangle \\
&= \frac{1}{N}\sum_{m} e^{i\tau m(\mu-\nu)}|0\rangle \\
&= \delta_{\nu\mu}|0\rangle.
\end{aligned}$$

Similarly, for a "time-bin" two-photons state we have

$$\hat{b}_\gamma|\tilde{1}_\mu,\tilde{1}_\nu\rangle = (\delta_{\gamma\mu}+\delta_{\gamma\nu})\,|0\rangle. \tag{G.11}$$

G.2.5 Calc for number operator on non localized time-bin single photon

Using the results from the previous section G.2.4, is straightforward to compute the expectation value of the number operator on a single photon state:

$$
\begin{aligned}
n(t) &= \langle \tilde{1}_\mu | \hat{n}(t) | \tilde{1}_\mu \rangle \\
&= \langle \tilde{1}_\mu | \hat{a}^\dagger(t)\, \hat{a}(t) | \tilde{1}_\mu \rangle \\
&= \langle \tilde{1}_\mu | \left(\sum_{\mu'} g^*_{\mu'}(t)\ \hat{b}^\dagger_{\mu'} \right) \left(\sum_{\nu'} g_{\nu'}(t)\ \hat{b}_{\nu'} \right) | \tilde{1}_\mu \rangle \\
&= \left(\sum_{\mu'} \ \langle \tilde{1}_\mu | g^*_{\mu'}(t)\ \hat{b}^\dagger_{\mu'} \right) \left(\sum_{\nu'} g_{\nu'}(t)\ \hat{b}_{\nu'} | \tilde{1}_\mu \rangle \right) \\
&= \left(\sum_{\mu'} \langle 0 | \delta_{\mu'\mu}\ g^*_{\mu'}(t) \right) \left(\sum_{\nu'} g_{\nu'}(t) \delta_{\nu'\mu} | 0 \rangle \right) \\
&= g^*_\mu(t)\ g_\mu(t)\ \ \langle 0 | 0 \rangle \\
&= |g_\mu(t)|^2
\end{aligned}
$$

summarizing:

$$
\langle \tilde{1}_\mu | \hat{n}(t) | \tilde{1}_\mu \rangle = |g_\mu(t)|^2. \tag{G.12}
$$

G.2.6 Calc. for number operator on non localized time-bin two photons

The expectation value of the number operator (mean photon number) on a state with two localized photons in two different time-bins is:

$$
\begin{aligned}
n(t) &= \langle 1_\mu, 1_\nu|\hat{n}(t)|1_\mu, 1_\nu\rangle \\
&= \langle 1_\mu, 1_\nu|\hat{a}^\dagger(t)\,\hat{a}(t)|1_\mu, 1_\nu\rangle \\
&= \langle 1_\mu, 1_\nu|\left(\sum_{\mu'} g^*_{\mu'}(t)\ \hat{b}^\dagger_{\mu'}\right)\left(\sum_{\nu'} g_{\nu'}(t)\ \hat{b}_{\nu'}\right)|1_\mu, 1_\nu\rangle \\
&= \left(\sum_{\mu'}\langle 1_\mu, 1_\nu|g^*_{\mu'}(t)\ \hat{b}^\dagger_{\mu'}\right)\left(\sum_{\nu'} g_{\nu'}(t)\ \hat{b}_{\nu'}|1_\mu, 1_\nu\rangle\right) \\
&= \left[\sum_{\mu'}(\langle 1_\nu|\delta_{\mu'\mu} + \langle 1_\mu|\delta_{\mu'\nu})g^*_{\mu'}(t)\right]\left[\sum_{\nu'} g_{\nu'}(t)\ \ (\delta_{\nu'\mu}|1_\nu\rangle + \delta_{\nu'\nu}|1_\mu)\right] \\
&= \left[\langle 1_\nu|g^*_\mu(t) + \langle 1_\mu|g^*_\nu(t)\right]\left[g_\mu(t)|1_\nu\rangle + g_\nu(t)|1_\mu\rangle\right] \\
&= \langle 1_\nu|1_\nu\rangle g^*_\mu(t)g_\mu(t) + \langle 1_\nu|1_\mu\rangle g^*_\mu(t)g_\nu(t) + \langle 1_\mu|1_\nu\rangle g^*_\nu(t)g_\mu(t) + \langle 1_\mu|1_\mu\rangle g^*_\nu(t)g_\nu(t) \\
&= |g_\mu(t)|^2 + |g_\nu(t)|^2
\end{aligned}
$$

summarizing

$$
\langle 1_\mu, 1_\nu|\hat{n}(t)|1_\mu, 1_\nu\rangle = |g_\mu(t)|^2 + |g_\nu(t)|^2 \tag{G.13}
$$

G.2.7 calculations for photon number on single-photon localized time-bin state

$$\langle\tilde{1};\alpha|\hat{n}(t)|\tilde{1};\alpha\rangle = \langle\tilde{1};\alpha|\hat{a}^\dagger(t)\,\hat{a}(t)|\tilde{1};\alpha\rangle \tag{G.14}$$

$$= \left(\sum_{\mu}\alpha^*_\mu\langle\tilde{1}_\mu|\right)\left(\sum_{\mu'} g^*_{\mu'}(t)\,\hat{b}^\dagger_{\mu'}\right)\left(\sum_{\nu'} g^*_{\nu'}(t)\,\hat{b}_{\nu'}\right)\left(\sum_{\nu}\alpha_\nu|\tilde{1}_\nu\rangle\right) \tag{G.15}$$

$$= \left(\sum_{\mu\,\mu'}\alpha^*_\mu\, g^*_{\mu'}(t)\,\langle\tilde{1}_\mu|\hat{b}^\dagger_{\mu'}\right)\left(\sum_{\nu\,\nu'} g^*_{\nu'}(t)\,\alpha_\nu\,\hat{b}_{\nu'}|\tilde{1}_\nu\rangle\right) \tag{G.16}$$

$$= \left(\sum_{\mu\,\mu'}\alpha^*_\mu\, g^*_{\mu'}(t)\,\langle 0|\delta_{\mu\,\mu'}\right)\left(\sum_{\nu\,\nu'} g^*_{\nu'}(t)\,\alpha_\nu\,\delta_{\nu\,\nu'}|0\rangle\right) \tag{G.17}$$

$$= \langle 0|\left(\sum_{\mu}\alpha^*_\mu\, g^*_{\mu}(t)\right)\left(\sum_{\nu} g^*_{\nu}(t)\,\alpha_\nu\right)|0\rangle \tag{G.18}$$

$$= \left(\sum_{\mu\,\nu}\alpha^*_\mu\, g^*_{\mu}(t) g^*_{\nu}(t)\,\alpha_\nu\right)\langle 0|0\rangle \tag{G.19}$$

$$= \sum_{\mu\,\nu}\alpha^*_\mu\,\alpha_\nu\, g^*_{\mu}(t) g^*_{\nu}(t) \tag{G.20}$$

G.2.8 calculations for photon number on two photon localized time-bin state

$$\langle\tilde{2};\alpha\,\beta|\hat{n}(t)|\tilde{2};\alpha\,\beta\rangle = \tag{G.21}$$

$$= \left(\sum_{\lambda\,\varepsilon}\alpha^*_\lambda\,\beta^*_\varepsilon\langle 1_\lambda, 1_\varepsilon|\right)\left(\sum_{\lambda'} g^*_{\lambda'}\,\hat{b}^\dagger_{\lambda'}\right)\left(\sum_{\mu'} g_{\mu'}\,\hat{b}_{\mu'}\right)\left(\sum_{\mu\,\nu}\alpha_\mu\beta_\nu|1_\mu, 1_\nu\rangle\right) \tag{G.22}$$

$$= \left(\sum_{\lambda\,\lambda'\,\varepsilon}\alpha^*_\lambda\,\beta^*_\varepsilon\, g^*_{\lambda'}\langle 1_\lambda, 1_\varepsilon|\,\hat{b}^\dagger_{\lambda'}\right)\left(\sum_{\mu\,\mu'\,\nu} g^*_{\mu'}\,\alpha_\mu\beta_\nu\,\hat{b}^\dagger_{\mu'}|1_\mu, 1_\nu\rangle\right) \tag{G.23}$$

$$= \left[\sum_{\lambda\,\lambda'\,\varepsilon}\alpha^*_\lambda\,\beta^*_\varepsilon\, g^*_{\lambda'}\left(\langle 1_\varepsilon|\,\delta_{\lambda\lambda'} + \langle 1_\lambda|\,\delta_{\varepsilon\lambda'}\right)\right]$$
$$\left[\sum_{\mu\,\mu'\,\nu} g_{\mu'}\,\alpha_\mu\beta_\nu\,\left(\delta_{\mu\mu'}|1_\nu\rangle + \delta_{\nu\mu'}|1_\mu\rangle\right)\right] \tag{G.24}$$

then, summing over λ' and μ'

$$= \left[\sum_{\lambda\,\lambda'\,\varepsilon} \alpha_\lambda^*\,\beta_\varepsilon^*\,(g_\lambda^*\langle 1_\varepsilon| + g_\varepsilon^*\langle 1_\lambda|)\right]\left[\sum_{\mu\,\mu'\,\nu} \alpha_\mu \beta_\nu\;(g_\mu|1_\nu\rangle + g_\nu|1_\mu\rangle)\right] \tag{G.25}$$

$$= \sum_{\lambda\,\varepsilon\,\mu\,\nu} \alpha_\lambda^*\,\beta_\varepsilon^*\,\alpha_\mu\,\beta_\nu\,(g_\lambda^*\,g_\mu\,\langle 1_\varepsilon|1_\nu\rangle + g_\lambda^*\,g_\nu\,\langle 1_\varepsilon|1_\mu\rangle + + g_\varepsilon^*\,g_\mu\,\langle 1_\lambda|1_\nu\rangle + g_\varepsilon^*\,g_\nu\,\langle 1_\lambda|1_\mu\rangle) \tag{G.26}$$

$$= \sum_{\lambda\,\varepsilon\,\mu\,\nu} \alpha_\lambda^*\,\beta_\varepsilon^*\,\alpha_\mu\,\beta_\nu\,(g_\lambda^*\,g_\mu\,\delta_{\varepsilon\nu} + g_\lambda^*\,g_\nu\,\delta_{\varepsilon\mu} + g_\varepsilon^*\,g_\mu\,\delta_{\lambda\nu} + g_\varepsilon^*\,g_\nu\,\delta_{\lambda\mu}) \tag{G.27}$$

$$= \sum_{\lambda\,\varepsilon\,\mu\,\nu} \alpha_\lambda^*\,\beta_\varepsilon^*\,\alpha_\mu\,\beta_\nu\,g_\lambda^*\,g_\mu\,\delta_{\varepsilon\nu} + \sum_{\lambda\,\varepsilon\,\mu\,\nu} \alpha_\lambda^*\,\beta_\varepsilon^*\,\alpha_\mu\,\beta_\nu\,g_\lambda^*\,g_\nu\,\delta_{\varepsilon\mu} + + \sum_{\lambda\,\varepsilon\,\mu\,\nu} \alpha_\lambda^*\,\beta_\varepsilon^*\,\alpha_\mu\,\beta_\nu\,g_\varepsilon^*\,g_\mu\,\delta_{\lambda\nu} + \sum_{\lambda\,\varepsilon\,\mu\,\nu} \alpha_\lambda^*\,\beta_\varepsilon^*\,\alpha_\mu\,\beta_\nu\,g_\varepsilon^*\,g_\nu\,\delta_{\lambda\mu} \tag{G.28}$$

[...]

G.2.9 Two time correlation function on two photons time-bins state

Let's evaluate the expectation value of the two time correlation function

$$G^2(T) \equiv \langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+T)\hat{a}(t+T)\hat{a}(t)\rangle; \tag{G.29}$$

onto the two time-bin localized photons state in (7.14)

$$|1_\mu, 1_\nu\rangle = \hat{b}_\mu^\dagger \hat{b}_\nu^\dagger |0,0\rangle \qquad \mu \neq \nu. \tag{G.30}$$

Now let's express the "time-dependant creation/annihilation operators" $\hat{a}^\dagger(t)/\hat{a}(t)$ in terms of "time-bin operators" (cfr. (A.2)):

$$\hat{a}(t) = \sum_{\mu=1}^{N} g_\mu(t)\,\hat{b}_\mu$$

$$
\begin{aligned}
G^2(T) &= \langle 1_\mu, 1_\nu|\, \hat{a}^\dagger(t)\ \hat{a}^\dagger(t+T)\ \hat{a}(t+T)\ \hat{a}(t)\,|1_\mu, 1_\nu\rangle \\
&= \langle 1_\mu, 1_\nu| \left(\sum_{\nu'} g^*_{\nu'}(t)\ \hat{b}^\dagger_{\nu'} \right) \left(\sum_{\mu'} g^*_{\mu'}(t+T)\ \hat{b}^\dagger_{\mu'} \right) \\
&\qquad \left(\sum_{\nu''} g_{\nu''}(t+T)\ \hat{b}_{\nu''} \right) \left(\sum_{\mu''} g_{\mu''}(t)\ \hat{b}_{\mu''} \right) |1_\mu, 1_\nu\rangle \\
&= \sum_{\nu',\mu',\nu'',\mu''} g^*_{\nu'}(t)\ g^*_{\mu'}(t+T)\ g_{\nu''}(t+T)\ g_{\mu''}(t) \langle 1_\mu, 1_\nu|\hat{b}^\dagger_{\nu'}\hat{b}^\dagger_{\mu'}\hat{b}_{\nu''}\hat{b}_{\mu''}|1_\mu, 1_\nu\rangle \\
&= \sum_{\nu',\mu',\nu'',\mu''} g^*_{\nu'}(t)\ g^*_{\mu'}(t+T)\ g_{\nu''}(t+T)\ g_{\mu''}(t) \\
&\qquad \langle 1_\mu, 1_\nu|\hat{b}^\dagger_{\nu'}\hat{b}^\dagger_{\mu'}\hat{b}_{\nu''}\left[\delta_{\mu'',\mu}|1_\nu\rangle + \delta_{\mu'',\nu}|1_\mu\rangle\right] \\
&= \sum_{\nu',\mu',\nu''} g^*_{\nu'}(t)\ g^*_{\mu'}(t+T)\ g_{\nu''}(t+T) \langle 1_\mu, 1_\nu|\hat{b}^\dagger_{\nu'}\hat{b}^\dagger_{\mu'}\hat{b}_{\nu''} \\
&\qquad \left[g_\mu(t)|1_\nu\rangle + g_\nu(t)|1_\mu\rangle\right] \\
&= \sum_{\nu',\mu',\nu''} g^*_{\nu'}(t)\ g^*_{\mu'}(t+T)\ g_{\nu''}(t+T) \langle 1_\mu, 1_\nu|\hat{b}^\dagger_{\nu'}\hat{b}^\dagger_{\mu'} \\
&\qquad \left[g_\mu(t)\hat{b}_{\nu''}|1_\nu\rangle + g_\nu(t)\hat{b}_{\nu''}|1_\mu\rangle\right] \\
&= \sum_{\nu',\mu',\nu''} g^*_{\nu'}(t)\ g^*_{\mu'}(t+T)\ g_{\nu''}(t+T) \langle 1_\mu, 1_\nu|\hat{b}^\dagger_{\nu'}\hat{b}^\dagger_{\mu'} \\
&\qquad \left[g_\mu(t)\delta_{\nu''\nu}|0\rangle + g_\nu(t)\delta_{\nu''\mu}|0\rangle\right] \\
&= \sum_{\nu',\mu'} g^*_{\nu'}(t)\ g^*_{\mu'}(t+T) \langle 1_\mu, 1_\nu|\hat{b}^\dagger_{\nu'}\hat{b}^\dagger_{\mu'} \\
&\qquad \left[g_\mu(t)g_\nu(t+T) + g_\nu(t)g_\mu(t+T)\right]|0\rangle \\
&= \sum_{\nu',\mu'} g^*_{\nu'}(t)\ g^*_{\mu'}(t+T) \left[\langle 1_\mu|\delta_{\nu'\nu} + \langle 1_\nu|\delta_{\nu'\mu}\right] \hat{b}^\dagger_{\mu'} \\
&\qquad \left[g_\mu(t)g_\nu(t+T) + g_\nu(t)g_\mu(t+T)\right]|0\rangle \\
&= \sum_{\nu'} g^*_{\mu'}(t+T) \left[\langle 1_\mu|g^*_\nu(t) + \langle 1_\nu|g^*_\mu(t)\right] \hat{b}^\dagger_{\mu'} \\
&\qquad \left[g_\mu(t)g_\nu(t+T) + g_\nu(t)g_\mu(t+T)\right]|0\rangle \\
&= \sum_{\nu'} g^*_{\mu'}(t+T) \left[\langle 0|\delta_{\mu'\mu}\ g^*_\nu(t) + \langle 0|\delta_{\mu'\nu}\ g^*_\mu(t)\right] \\
&\qquad \left[g_\mu(t)g_\nu(t+T) + g_\nu(t)g_\mu(t+T)\right]|0\rangle \\
&= \left[g^*_\mu(t+T)\ g^*_\nu(t) + g^*_\nu(t+T)\ g^*_\mu(t)\right]\left[g_\mu(t)g_\nu(t+T) + g_\nu(t)g_\mu(t+T)\right]\langle 0|0\rangle \\
&= \left[g^*_\mu(t)\ g^*_\nu(t+T) + g^*_\nu(t)\ g^*_\mu(t+T)\right]\left[g_\mu(t)\ g_\nu(t+T) + g_\nu(t)g_\mu(t+T)\right]
\end{aligned}
$$

finally:

$$G^2(T) = |g_\mu(t)\; g_\nu(t+T) + g_\nu(t) g_\mu(t+T)|^2 \tag{G.31}$$

G.2.10 Hong-Ou-Mandel preliminaries

In this section we carry out a simpler calculation to introduce the one in the following section. Also, by now the aim is to obtain the same result as in formula (80) in [KNR+07]:

$$\langle\psi_{out}|\hat{C}|\psi_{out}\rangle = \frac{1}{2} - \frac{1}{2}\sum_{i,j}\alpha_i\alpha_j^*\beta_j\beta_i^*$$

so to better understand how to compute the more general case (the one with the delay).

First, let's consider the coincidence operator $\hat{C}$:

$$\hat{C} \equiv \hat{a}^\dagger(t)\; \hat{b}^\dagger(t)\; \hat{b}(t)\; \hat{a}(t)$$

then, let's express the *time dependent* creation/annihilation operators in terms of the *single mode* ones:

$$\hat{a}(t) = \sum_{n=1}^{\infty} \hat{a}_n\; e^{-i\,n\,t}$$

so that

$$\begin{aligned}\hat{C} &= \sum_{n=1}^{\infty}\hat{a}_n^\dagger\, e^{i\,n\,t} \sum_{m=1}^{\infty}\hat{b}_n^\dagger\, e^{-i\,m\,t} \sum_{n=1}^{\infty}\hat{b}_k\, e^{-i\,k\,t} \sum_{l=1}^{\infty}\hat{a}_l\, e^{-i\,l\,t} &\text{(G.32)}\\ &= \sum_{n,m,k,l=1}^{\infty} e^{i\,(n+m-k-l)\,t}\;\; \hat{a}_n^\dagger\, \hat{b}_m^\dagger\, \hat{b}_k\, \hat{a}_l &\text{(G.33)}\end{aligned}$$

Now, let's consider the expectation value of this coincidence operator on the state coming out from a beam splitter where two photons have entered the two different input ports:

$$
\begin{aligned}
|\psi_{out}\rangle &= \frac{1}{2}\sum_{i,j}^{\infty}\alpha_i\,\beta_j\ \left[|1_{a_i},1_{b_j}\rangle - |1_{a_i},1_{a_j},0_b\rangle + |0_a,1_{b_i},1_{b_j}\rangle - |1_{a_j},1_{b_i}\rangle\right] \\
&= \frac{1}{2}\sum_{i,j}^{\infty}\alpha_i\,\beta_j\ \left[\hat{a}_i^{\dagger}\hat{b}_j^{\dagger}|0\rangle - \hat{a}_i^{\dagger}\hat{a}_j^{\dagger}|0\rangle + \hat{b}_i^{\dagger}\hat{b}_j^{\dagger}|0\rangle - \hat{a}_j^{\dagger}\hat{b}_i^{\dagger}|0\rangle\right]
\end{aligned}
$$

$$
\langle\psi_{out}|\hat{C}|\psi_{out}\rangle = \tag{G.34}
$$

$$
= \left[\frac{1}{2}\sum_{i,j}^{\infty}\alpha_i^*\,\beta_j^*\ \left(\langle 1_{a_i},1_{b_j}| - \langle 1_{a_i},1_{a_j},0_b| + \langle 0_a,1_{b_i},1_{b_j}| - \langle 1_{a_j},1_{b_i}|\right)\right]
$$

$$
\hat{C}\left[\frac{1}{2}\sum_{i',j'}^{\infty}\alpha_{i'}\,\beta_{j'}\ \left(|1_{a_{i'}},1_{b_{j'}}\rangle - |1_{a_{i'}},1_{a_{j'}},0_b\rangle + |0_a,1_{b_{i'}},1_{b_{j'}}\rangle - |1_{a_{j'}},1_{b_{i'}}\rangle\right)\right] \tag{G.35}
$$

$$
= \frac{1}{2}\,\frac{1}{2}\sum_{i,j,i',j'}^{\infty}\alpha_i^*\,\beta_j^*\,\alpha_{i'}\,\beta_{j'}\left[\langle 1_{a_i},1_{b_j}| - \langle 1_{a_i},1_{a_j},0_b| + \langle 0_a,1_{b_i},1_{b_j}| - \langle 1_{a_j},1_{b_i}|\right]
$$

$$
\hat{C}\left[|1_{a_{i'}},1_{b_{j'}}\rangle - |1_{a_{i'}},1_{a_{j'}},0_b\rangle + |0_a,1_{b_{i'}},1_{b_{j'}}\rangle - |1_{a_{j'}},1_{b_{i'}}\rangle\right] \tag{G.36}
$$

now we express the coincidence operator, in terms of creation/annihilation operators:

$$
\begin{aligned}
&= \frac{1}{4} \sum_{i,j,i',j'}^{\infty} \alpha_i^* \, \beta_j^* \, \alpha_{i'} \, \beta_{j'} \, [\langle 1_{a\,i}, 1_{b\,j}| - \langle 1_{a\,i}, 1_{a\,j}| + \langle 1_{b\,i}, 1_{b\,j}| - \langle 1_{a\,j}, 1_{b\,i}|] \\
&\sum_{n,m,k,l=1}^{\infty} e^{i\,(n+m-k-l)\,t} \quad \hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \\
&[|1_{a\,i'}, 1_{b\,j'}\rangle - |1_{a\,i'}, 1_{a\,j'}\rangle + |1_{b\,i'}, 1_{b\,j'}\rangle - |1_{a\,j'}, 1_{b\,i'}\rangle] \qquad \text{(G.37)} \\
&= \frac{1}{4} \sum_{i,j,i',j',n,m,k,l}^{\infty} e^{i\,(n+m-k-l)\,t} \; \alpha_i^* \, \beta_j^* \, \alpha_{i'} \, \beta_{j'} \\
&[\langle 1_{a\,i}, 1_{b\,j}| - \langle 1_{a\,i}, 1_{a\,j}| + \langle 1_{b\,i}, 1_{b\,j}| - \langle 1_{a\,j}, 1_{b\,i}|] \\
&\hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \\
&[|1_{a\,i'}, 1_{b\,j'}\rangle - |1_{a\,i'}, 1_{a\,j'}\rangle + |1_{b\,i'}, 1_{b\,j'}\rangle - |1_{a\,j'}, 1_{b\,i'}\rangle] \qquad \text{(G.38)} \\
&= \frac{1}{4} \sum_{i,j,i',j',n,m,k,l}^{\infty} e^{i\,(n+m-k-l)\,t} \; \alpha_i^* \, \beta_j^* \, \alpha_{i'} \, \beta_{j'} \\
&\Big[\langle 1_{a\,i}, 1_{b\,j}| \;\; \hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \;\; |1_{a\,i'}, 1_{b\,j'}\rangle - \langle 1_{a\,i}, 1_{b\,j}| \;\; \hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \;\; |1_{b\,i'}, 1_{b\,j'}\rangle + \\
&+ \langle 1_{a\,i}, 1_{b\,j}| \;\; \hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \;\; |1_{a\,i'}, 1_{a\,j'}\rangle - \langle 1_{a\,i}, 1_{b\,j}| \;\; \hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \;\; |1_{a\,j'}, 1_{b\,i'}\rangle - \\
&- \langle 1_{a\,i}, 1_{a\,j}| \;\; \hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \;\; |1_{a\,i'}, 1_{b\,j'}\rangle + \langle 1_{a\,i}, 1_{a\,j}| \;\; \hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \;\; |1_{a\,i'}, 1_{a\,j'}\rangle - \\
&- \langle 1_{a\,i}, 1_{a\,j}| \;\; \hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \;\; |1_{b\,i'}, 1_{b\,j'}\rangle + \langle 1_{a\,i}, 1_{a\,j}| \;\; \hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \;\; |1_{a\,j'}, 1_{b\,i'}\rangle + \\
&+ \langle 1_{b\,i}, 1_{b\,j}| \;\; \hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \;\; |1_{a\,i'}, 1_{b\,j'}\rangle - \langle 1_{b\,i}, 1_{b\,j}| \;\; \hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \;\; |1_{a\,i'}, 1_{a\,j'}\rangle + \\
&+ \langle 1_{b\,i}, 1_{b\,j}| \;\; \hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \;\; |1_{b\,i'}, 1_{b\,j'}\rangle - \langle 1_{b\,i}, 1_{b\,j}| \;\; \hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \;\; |1_{a\,j'}, 1_{b\,i'}\rangle - \\
&- \langle 1_{a\,j}, 1_{b\,i}| \;\; \hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \;\; |1_{a\,i'}, 1_{b\,j'}\rangle + \langle 1_{a\,j}, 1_{b\,i}| \;\; \hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \;\; |1_{a\,i'}, 1_{a\,j'}\rangle - \\
&- \langle 1_{a\,j}, 1_{b\,i}| \;\; \hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \;\; |1_{b\,i'}, 1_{b\,j'}\rangle + \langle 1_{a\,j}, 1_{b\,i}| \;\; \hat{a}_n^\dagger \, \hat{b}_m^\dagger \, \hat{b}_k \, \hat{a}_l \;\; |1_{a\,j'}, 1_{b\,i'}\rangle \Big] \qquad \text{(G.39)}
\end{aligned}
$$

Here we have $4 \times 4 = 16$ terms, and I'll compute each of them separately.

To compute each term we apply the operators to the states, using the result $\hat{a}_i^\dagger |1_{a_j}\rangle = \delta_{i,j}|0\rangle$, and then calculating the sums. Because of the deltas, only some terms will survive.

term 1

$$\langle 1_{a\,i}, 1_{b\,j}|\hat{a}_n^\dagger\,\hat{b}_m^\dagger\,\hat{b}_k\,\hat{a}_l|1_{a\,i'}, 1_{b\,j'}\rangle = \quad (G.40)$$

$$\langle 1_{b\,j}|\delta_{i\,n}\;\hat{b}_m^\dagger\,\hat{b}_k\,\delta_{l\,i'}|1_{b\,j'}\rangle = \quad (G.41)$$

$$= \delta_{in}\,\delta_{jm}\,\delta_{kj'}\,\delta_{li'}\langle 0|0\rangle \quad (G.42)$$

$$= \delta_{in}\,\delta_{jm}\,\delta_{kj'}\,\delta_{li'} \quad (G.43)$$

term 2

$$\langle 0|\hat{a}_i\,\hat{b}_j\quad \hat{a}_n^\dagger\,\hat{b}_m^\dagger\,\hat{b}_k\,\hat{a}_l\quad \hat{b}_{i'}^\dagger\,\hat{b}_{j'}^\dagger|0\rangle \quad (G.44)$$

$$= \langle 1_{ai}, 1_{bj}|\hat{a}_n^\dagger\,\hat{b}_m^\dagger\,\hat{b}_k\,\hat{a}_l\;|0_a, 1_{bi}, 1_{bj}\rangle = 0 \quad (G.45)$$

it's zero since on the right there is the annihilation operator $\hat{a}_l$ acting on the vacuum $|0_a\rangle$.

term 4

$$\langle 1_{ai}, 1_{bj}|\hat{a}_n^\dagger\,\hat{b}_m^\dagger\,\hat{b}_k\,\hat{a}_l|1_{aj'}, 1_{bi'}\rangle \quad (G.46)$$

$$= \langle 1_{bj}|\delta_{in}\,\hat{b}_m^\dagger\,\hat{b}_k\,\delta_{lj'}|1_{bi'}\rangle \quad (G.47)$$

$$= \delta_{in}\,\delta_{jm}\,\delta_{ki'}\,\delta_{lj'}\langle 0|0\rangle \quad (G.48)$$

$$= \delta_{in}\,\delta_{jm}\,\delta_{ki'}\,\delta_{lj'}. \quad (G.49)$$

Now we have seen a general criterion: the only non-zero terms are those that have both spatial modes on both sides. These terms are 1, 4, 13 and 16:

term 13

$$\langle 0|\hat{a}_j\,\hat{b}_i\quad \hat{a}_n^\dagger\,\hat{b}_m^\dagger\,\hat{b}_k\,\hat{a}_l\quad \hat{a}_{i'}^\dagger\,\hat{b}_{j'}^\dagger|0\rangle \quad (G.50)$$

$$= \langle 1_{aj}, 1_{bi}|\hat{a}_n^\dagger\,\hat{b}_m^\dagger\,\hat{b}_k\,\hat{a}_l|1_{ai'}, 1_{bj'}\rangle \quad (G.51)$$

$$= \delta_{jn}\;\delta_{im}\;\delta_{kj'}\;\delta_{li'} \quad (G.52)$$

term 16

$$\langle 0|\hat{a}_j\,\hat{b}_i \quad \hat{a}_n^\dagger\,\hat{b}_m^\dagger\,\hat{b}_k\,\hat{a}_l \quad \hat{a}_{j'}^\dagger\,\hat{b}_{i'}^\dagger|0\rangle \tag{G.53}$$
$$= \langle 1_{aj}, 1_{bi}|\hat{a}_n^\dagger\,\hat{b}_m^\dagger\,\hat{b}_k\,\hat{a}_l|1_{aj'}, 1_{bi'}\rangle \tag{G.54}$$
$$= \delta_{jn}\;\delta_{im}\;\delta_{ki'}\;\delta_{lj'} \tag{G.55}$$

Summarizing, from (G.39) we have:

$$\langle\psi_{out}|\hat{C}|\psi_{out}\rangle = \tag{G.56}$$
$$= \frac{1}{4}\sum_{i,j,i',j',n,m,k,l}^{\infty} e^{i\,(n+m-k-l)\,t}\;\alpha_i^*\,\beta_j^*\,\alpha_{i'}\,\beta_{j'}$$
$$[\delta_{in}\,\delta_{jm}\,\delta_{kj'}\,\delta_{li'} - \delta_{in}\,\delta_{jm}\,\delta_{ki'}\,\delta_{lj'} - \delta_{jn}\;\delta_{im}\;\delta_{kj'}\;\delta_{li'} + \delta_{jn}\;\delta_{im}\;\delta_{ki'}\;\delta_{lj'}] \tag{G.57}$$
$$= \frac{1}{4}\sum_{i,j,i',j',n,m,k,l}^{\infty} e^{i\,(n+m-k-l)\,t}\;\alpha_i^*\,\beta_j^*\,\alpha_{i'}\,\beta_{j'}\;\delta_{in}\,\delta_{jm}\,\delta_{kj'}\,\delta_{li'}$$
$$-\frac{1}{4}\sum_{i,j,i',j',n,m,k,l}^{\infty} e^{i\,(n+m-k-l)\,t}\;\alpha_i^*\,\beta_j^*\,\alpha_{i'}\,\beta_{j'}\;\delta_{in}\,\delta_{jm}\,\delta_{ki'}\,\delta_{lj'}$$
$$-\frac{1}{4}\sum_{i,j,i',j',n,m,k,l}^{\infty} e^{i\,(n+m-k-l)\,t}\;\alpha_i^*\,\beta_j^*\,\alpha_{i'}\,\beta_{j'}\;\delta_{jn}\;\delta_{im}\;\delta_{kj'}\;\delta_{li'}$$
$$+\frac{1}{4}\sum_{i,j,i',j',n,m,k,l}^{\infty} e^{i\,(n+m-k-l)\,t}\;\alpha_i^*\,\beta_j^*\,\alpha_{i'}\,\beta_{j'}\;\delta_{jn}\;\delta_{im}\;\delta_{ki'}\;\delta_{lj'} \tag{G.58}$$
$$= \frac{1}{4}\sum_{n,m,k,l}^{\infty} e^{i\,(n+m-k-l)\,t}\;\alpha_n^*\,\beta_m^*\,\alpha_l\,\beta_k - \frac{1}{4}\sum_{n,m,k,l}^{\infty} e^{i\,(n+m-k-l)\,t}\;\alpha_n^*\,\beta_m^*\,\alpha_k\,\beta_l$$
$$-\frac{1}{4}\sum_{n,m,k,l}^{\infty} e^{i\,(n+m-k-l)\,t}\;\alpha_m^*\,\beta_n^*\,\alpha_l\,\beta_k + \frac{1}{4}\sum_{n,m,k,l}^{\infty} e^{i\,(n+m-k-l)\,t}\;\alpha_m^*\,\beta_n^*\,\alpha_k\,\beta_l \tag{G.59}$$

Marcin recalculation of HOM preliminaries

The idea by Marcin is about a different way of writing the coincidence operator $\hat{C} = \hat{a}^\dagger(t)\,\hat{b}^\dagger(t)\,\hat{b}(t)\,\hat{a}(t)$, and it consist in writing it in terms of number operators $\hat{C} = \hat{n}_a(t)\,\hat{n}_b(t)$:

$$\langle\psi_{out}|\hat{C}|\psi_{out}\rangle = \tag{G.60}$$

$$= \frac{1}{2}\,\frac{1}{2}\sum_{i,j,i',j'}^{\infty} \alpha_i^*\,\beta_j^*\,\alpha_{i'}\,\beta_{j'}\,\left[\langle 1_{a_i},1_{b_j}|-\langle 1_{a_i},1_{a_j},0_b|+\langle 0_a,1_{b_i},1_{b_j}|-\langle 1_{a_j},1_{b_i}|\right]$$

$$\hat{n}_a(t)\,\hat{n}_b(t)\left[|1_{a_{i'}},1_{b_{j'}}\rangle - |1_{a_{i'}},1_{a_{j'}},0_b\rangle + |0_a,1_{b_{i'}},1_{b_{j'}}\rangle - |1_{a_{j'}},1_{b_{i'}}\rangle\right] \tag{G.61}$$

$$= \frac{1}{4}\sum_{i,j,i',j',n,m}^{\infty} \alpha_i^*\,\beta_j^*\,\alpha_{i'}\,\beta_{j'}$$

$$[\langle 1_{a\,i},1_{b\,j}|-\langle 1_{a\,i},1_{a\,j}|+\langle 1_{b\,i},1_{b\,j}|-\langle 1_{a\,j},1_{b\,i}|]$$

$$e^{i\,n\,t}\hat{a}_n^\dagger e^{-i\,n\,t}\,\hat{a}_n\;e^{i\,m\,t}\,\hat{b}_m^\dagger\;e^{-i\,m\,t}\,\hat{b}_m$$

$$[|1_{a\,i'},1_{b\,j'}\rangle - |1_{a\,i'},1_{a\,j'}\rangle + |1_{b\,i'},1_{b\,j'}\rangle - |1_{a\,j'},1_{b\,i'}\rangle] \tag{G.62}$$

$$= \frac{1}{4}\sum_{i,j,i',j',n,m}^{\infty} \alpha_i^*\,\beta_j^*\,\alpha_{i'}\,\beta_{j'}$$

$$\Big[\langle 1_{a\,i},1_{b\,j}|\;\;\hat{a}_n^\dagger\,\hat{a}_n\,\hat{b}_m^\dagger\,\hat{b}_m\;\;|1_{a\,i'},1_{b\,j'}\rangle - \langle 1_{a\,i},1_{b\,j}|\;\;\hat{a}_n^\dagger\,\hat{a}_n\,\hat{b}_m^\dagger\,\hat{b}_m\;\;|1_{b\,i'},1_{b\,j'}\rangle +$$

$$+\langle 1_{a\,i},1_{b\,j}|\;\;\hat{a}_n^\dagger\,\hat{a}_n\,\hat{b}_m^\dagger\,\hat{b}_m\;\;|1_{a\,i'},1_{a\,j'}\rangle - \langle 1_{a\,i},1_{b\,j}|\;\;\hat{a}_n^\dagger\,\hat{a}_n\,\hat{b}_m^\dagger\,\hat{b}_m\;\;|1_{a\,j'},1_{b\,i'}\rangle -$$

$$-\langle 1_{a\,i},1_{a\,j}|\;\;\hat{a}_n^\dagger\,\hat{a}_n\,\hat{b}_m^\dagger\,\hat{b}_m\;\;|1_{a\,i'},1_{b\,j'}\rangle + \langle 1_{a\,i},1_{a\,j}|\;\;\hat{a}_n^\dagger\,\hat{a}_n\,\hat{b}_m^\dagger\,\hat{b}_m\;\;|1_{a\,i'},1_{a\,j'}\rangle -$$

$$-\langle 1_{a\,i},1_{a\,j}|\;\;\hat{a}_n^\dagger\,\hat{a}_n\,\hat{b}_m^\dagger\,\hat{b}_m\;\;|1_{b\,i'},1_{b\,j'}\rangle + \langle 1_{a\,i},1_{a\,j}|\;\;\hat{a}_n^\dagger\,\hat{a}_n\,\hat{b}_m^\dagger\,\hat{b}_m\;\;|1_{a\,j'},1_{b\,i'}\rangle +$$

$$+\langle 1_{b\,i},1_{b\,j}|\;\;\hat{a}_n^\dagger\,\hat{a}_n\,\hat{b}_m^\dagger\,\hat{b}_m\;\;|1_{a\,i'},1_{b\,j'}\rangle - \langle 1_{b\,i},1_{b\,j}|\;\;\hat{a}_n^\dagger\,\hat{a}_n\,\hat{b}_m^\dagger\,\hat{b}_m\;\;|1_{a\,i'},1_{a\,j'}\rangle +$$

$$+\langle 1_{b\,i},1_{b\,j}|\;\;\hat{a}_n^\dagger\,\hat{a}_n\,\hat{b}_m^\dagger\,\hat{b}_m\;\;|1_{b\,i'},1_{b\,j'}\rangle - \langle 1_{b\,i},1_{b\,j}|\;\;\hat{a}_n^\dagger\,\hat{a}_n\,\hat{b}_m^\dagger\,\hat{b}_m\;\;|1_{a\,j'},1_{b\,i'}\rangle -$$

$$-\langle 1_{a\,j},1_{b\,i}|\;\;\hat{a}_n^\dagger\,\hat{a}_n\,\hat{b}_m^\dagger\,\hat{b}_m\;\;|1_{a\,i'},1_{b\,j'}\rangle + \langle 1_{a\,j},1_{b\,i}|\;\;\hat{a}_n^\dagger\,\hat{a}_n\,\hat{b}_m^\dagger\,\hat{b}_m\;\;|1_{a\,i'},1_{a\,j'}\rangle -$$

$$-\;\langle 1_{a\,j},1_{b\,i}|\;\;\hat{a}_n^\dagger\,\hat{a}_n\,\hat{b}_m^\dagger\,\hat{b}_m\;\;|1_{b\,i'},1_{b\,j'}\rangle + \langle 1_{a\,j},1_{b\,i}|\;\;\hat{a}_n^\dagger\,\hat{a}_n\,\hat{b}_m^\dagger\,\hat{b}_m\;\;|1_{a\,j'},1_{b\,i'}\rangle\Big] \tag{G.63}$$

Then there are the 16 terms computed separately:

term 1

$$\langle 1_{a\,i},1_{b\,j}|\hat{a}_n^\dagger\,\hat{a}_n\,\hat{b}_m^\dagger\,\hat{b}_m|1_{a\,i'},1_{b\,j'}\rangle = \tag{G.64}$$

$$= \langle 1_{b\,j}|\delta_{i\,n}\;\hat{b}_m^\dagger\,\hat{b}_m\,\delta_{n\,i'}|1_{b\,j'}\rangle \tag{G.65}$$

$$= \delta_{in}\,\delta_{jm}\,\delta_{mj'}\,\delta_{ni'}\langle 0|0\rangle \tag{G.66}$$

$$= \delta_{in}\,\delta_{jm}\,\delta_{mj'}\,\delta_{ni'} \tag{G.67}$$

where we have used the normality of number states, so that $\langle 0|0\rangle = 1$.

term 4

$$\langle 1_{ai}, 1_{bj}|\hat{a}_n^\dagger\, \hat{b}_m^\dagger\, \hat{b}_m\, \hat{a}_n|1_{aj'}, 1_{bi'}\rangle \tag{G.68}$$

$$= \delta_{in}\, \delta_{jm}\, \delta_{mi'}\, \delta_{nj'} \tag{G.69}$$

term 13

$$\langle 0|\hat{a}_j\, \hat{b}_i \quad \hat{a}_n^\dagger\, \hat{b}_m^\dagger\, \hat{b}_m\, \hat{a}_n \quad \hat{a}_{i'}^\dagger\, \hat{b}_{j'}^\dagger|0\rangle \tag{G.70}$$

$$= \delta_{jn}\ \delta_{im}\ \delta_{mj'}\ \delta_{ni'} \tag{G.71}$$

term 16

$$\langle 0|\hat{a}_j\, \hat{b}_i \quad \hat{a}_n^\dagger\, \hat{b}_m^\dagger\, \hat{b}_m\, \hat{a}_n \quad \hat{a}_{j'}^\dagger\, \hat{b}_{i'}^\dagger|0\rangle \tag{G.72}$$

$$= \delta_{jn}\ \delta_{im}\ \delta_{mi'}\ \delta_{nj'} \tag{G.73}$$

Summarizing, from (G.63) we have:

$$\langle\psi_{out}|\hat{C}|\psi_{out}\rangle = \tag{G.74}$$

$$= \frac{1}{4} \sum_{i,j,i',j',n,m}^{\infty} \alpha_i^*\, \beta_j^*\, \alpha_{i'}\, \beta_{j'}$$
$$[\delta_{in}\, \delta_{jm}\, \delta_{mj'}\, \delta_{ni'} - \delta_{in}\, \delta_{jm}\, \delta_{mi'}\, \delta_{nj'} - \delta_{jn}\ \delta_{im}\ \delta_{mj'}\ \delta_{ni'} + \delta_{jn}\ \delta_{im}\ \delta_{mi'}\ \delta_{nj'}] \tag{G.75}$$

then we sum over m and n and use the relation $\sum_n \delta_{i\,n}\delta_{n\,i'} = \delta_{i\,i'}$:

$$
\begin{aligned}
&= \frac{1}{4}\sum_{i,j,i',j'}^{\infty} \alpha_i^*\,\beta_j^*\,\alpha_{i'}\,\beta_{j'}\;\delta_{i\,i'}\delta_{j\,j'} - \frac{1}{4}\sum_{i,j,i',j'}^{\infty} \alpha_i^*\,\beta_j^*\,\alpha_{i'}\,\beta_{j'}\;\delta_{i\,j'}\,\delta_{j\,i'} \\
&-\frac{1}{4}\sum_{i,j,i',j'}^{\infty} \alpha_i^*\,\beta_j^*\,\alpha_{i'}\,\beta_{j'}\;\delta_{j\,i'}\delta_{i\,j'} + \frac{1}{4}\sum_{i,j,i',j'}^{\infty} \alpha_i^*\,\beta_j^*\,\alpha_{i'}\,\beta_{j'}\;\delta_{j\,j'}\,\delta_{i\,i'} \qquad \text{(G.76)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{i\,j}^{\infty} \alpha_i^*\,\beta_j^*\,\alpha_i\,\beta_j - \frac{1}{4}\sum_{i\,j}^{\infty} \alpha_i^*\,\beta_j^*\,\alpha_j\,\beta_i - \\
&\qquad -\frac{1}{4}\sum_{i\,j}^{\infty} \alpha_i^*\,\beta_j^*\,\alpha_j\,\beta_i + \frac{1}{4}\sum_{i\,j}^{\infty} \alpha_i^*\,\beta_j^*\,\alpha_i\,\beta_j \qquad \text{(G.77)}
\end{aligned}
$$

$$
= \frac{1}{2}\sum_{i\,j}^{\infty} |\alpha_i|^2\,|\beta_j|^2 - \frac{1}{2}\sum_{i\,j}^{\infty} \alpha_i^*\,\beta_j^*\,\alpha_j\,\beta_i \qquad \text{(G.78)}
$$

$$
= \frac{1}{2} - \frac{1}{2}\sum_{i\,j}^{\infty} \alpha_i^*\,\beta_j^*\,\alpha_j\,\beta_i \qquad \text{(G.79)}
$$

where we have used the normalization relation $\sum_n |\alpha_n|^2 = \sum_m |\beta_m|^2 = 1$.

G.2.11 Hong-Ou-Mandel calculations

In this section we want again to compute the time average of the expectation value of the coincidence function on a state with two photons exiting from a beam splitter:

$$
\overline{\langle \psi_{out}|C|\psi_{out}\rangle}. \qquad \text{(G.80)}
$$

but this time we want to have *localized photons*, so we will consider a state built with *time-bin operators* acting on the vacuum. First, let's consider the case where the two measurements on the two output ports are performed at two different times separated by an interval T.

We use the definition of the coincidence function (7.38):

$$
C \equiv \langle \hat{c}^\dagger(t)\hat{d}^\dagger(t+T)\hat{d}(t+T)\hat{c}(t)\rangle \qquad \text{(G.81)}
$$

so that

$$\langle\psi_{out}|C|\psi_{out}\rangle = \langle\psi_{out}|\hat{c}^\dagger(t)\hat{d}^\dagger(t+T)\hat{d}(t+T)\hat{c}(t)|\psi_{out}\rangle \tag{G.82}$$

we also recall the expression (7.34) of the state exiting the beam splitter:

$$|\psi_{out}\rangle = \frac{1}{2}\sum_{\mu,\nu=1}^{\infty}\alpha_\mu\beta_\nu\left[|1_{c\mu},1_{c\nu},0_d\rangle - |1_{c\mu},1_{d\nu}\rangle + |1_{c\nu},1_{d\mu}\rangle - |0_c,1_{d\mu},1_{d\nu}\rangle\right].$$

If we write either the coincidence function and the output state in terms of time-bin operators we have:

$$\hat{c}^\dagger(t)\ \hat{d}^\dagger(t+T)\ \hat{d}(t+T)\ \hat{c}(t) = \sum_{\mu'\nu'\mu''\nu''=1}^{N} g^*_{\mu'}(t)\ g^*_{\nu'}(t+T)\ g_{\mu''}(t+T)\ g_{\nu''}(t)\ \ \hat{c}^\dagger_{\mu'}\ \hat{d}^\dagger_{\nu'}\ \hat{d}_{\mu''}\ \hat{c}_{\nu''} \tag{G.83}$$

(cfr. (G.8)), and

$$|\psi_{out}\rangle = \frac{1}{2}\sum_{\mu,\nu=1}^{\infty}\alpha_\mu\beta_\nu\left[\hat{c}^\dagger_\mu\hat{c}^\dagger_\nu|0\rangle - \hat{c}^\dagger_\mu\hat{d}^\dagger_\nu|0\rangle + \hat{c}^\dagger_\nu\hat{d}^\dagger_\mu|0\rangle - \hat{d}^\dagger_\mu\hat{d}^\dagger_\nu|0\rangle\right] \tag{G.84}$$

(cfr. (7.14)).

This recalled, implementing these expressions we have:

$$\langle\psi_{out}|C|\psi_{out}\rangle =$$

$$= \frac{1}{2}\sum_{\mu,\nu=1}^{\infty}\alpha_\mu\beta_\nu \sum_{\mu'\nu'\mu''\nu''=1}^{N} g^*_{\mu'}(t)g^*_{\nu'}(t+T)g_{\mu''}(t+T)g_{\nu''}(t)$$

$$\langle\psi_{out}|\hat{c}^\dagger_{\mu'}\hat{d}^\dagger_{\nu'}\hat{d}_{\mu''}\hat{c}_{\nu''}\left[|1_{c\mu},1_{c\nu},0_d\rangle - |1_{c\mu},1_{d\nu}\rangle + |1_{c\nu},1_{d\mu}\rangle - |0_c,1_{d\mu},1_{d\nu}\rangle\right] \quad \text{(G.85)}$$

$$= \quad '' \quad '' \quad ''$$

$$\Big[\langle\psi_{out}|\hat{c}^\dagger_{\mu'}\hat{d}^\dagger_{\nu'}\hat{d}_{\mu''}\hat{c}_{\nu''}|1_{c\mu},1_{c\nu},0_d\rangle -$$

$$- \quad '' \quad |1_{c\mu},1_{d\nu}\rangle \quad +$$

$$+ \quad '' \quad |1_{c\nu},1_{d\mu}\rangle \quad -$$

$$- \quad '' \quad |0_c,1_{d\mu},1_{d\nu}\rangle] \quad \text{(G.86)}$$

$$= \quad '' \quad '' \quad '' \quad '' \quad ''$$

$$\Big[\langle\psi_{out}|\hat{c}^\dagger_{\mu'}\hat{d}^\dagger_{\nu'}\hat{d}_{\mu''}|0\rangle(\delta_{\nu''\mu}+\delta_{\nu''\nu}) -$$

$$- \quad '' \quad '' \quad \hat{d}_{\mu''}|1_{d\nu}\rangle \quad \delta_{\nu''\mu} \quad +$$

$$+ \quad '' \quad '' \quad \hat{d}_{\mu''}|1_{d\mu}\rangle \quad \delta_{\nu''\nu} \quad -$$

$$- \quad '' \quad '' \quad 0 \quad] \quad \text{(G.87)}$$

$$= \frac{1}{2}\sum_{\mu,\nu=1}^{\infty}\alpha_\mu\beta_\nu \sum_{\mu'\nu'\mu''\nu''=1}^{N} g^*_{\mu'}(t)\; g^*_{\nu'}(t+T)$$

$$\Big[g_{\mu''}(t+T)\; g_{\nu''}(t)\; \langle\psi_{out}|\hat{c}^\dagger_{\mu'}\hat{d}^\dagger_{\nu'}\hat{d}_{\mu''}|0\rangle(\delta_{\nu''\mu}+\delta_{\nu''\nu}) -$$

$$-g_{\mu''}(t+T)\; g_{\nu''}(t) \quad '' \quad '' \quad \hat{d}_{\mu''}|1_{d\nu}\rangle \quad \delta_{\nu''\mu} +$$

$$+g_{\mu''}(t+T)\; g_{\nu''}(t) \quad '' \quad '' \quad \hat{d}_{\mu''}|1_{d\mu}\rangle \quad \delta_{\nu''\nu}\Big] \quad \text{(G.88)}$$

now we compute the sum over ν''

$$= \frac{1}{2}\sum_{\mu,\nu=1}^{\infty}\alpha_\mu\beta_\nu \sum_{\mu'\nu'\mu''=1}^{N} g^*_{\mu'}(t)g^*_{\nu'}(t+T)$$

$$\Big[g_{\mu''}(t+T)g_\mu(t)\langle\psi_{out}|\hat{c}^\dagger_{\mu'}\hat{d}^\dagger_{\nu'}\hat{d}_{\mu''}|0\rangle +$$

$$+g_{\mu''}(t+T)g_\nu(t) \quad '' \quad '' \quad \hat{d}_{\mu''}|0\rangle -$$

$$-g_{\mu''}(t+T)g_\mu(t) \quad '' \quad '' \quad \hat{d}_{\mu''}|1_{d\nu}\rangle +$$

$$+g_{\mu''}(t+T)g_\nu(t) \quad '' \quad '' \quad \hat{d}_{\mu''}|1_{d\mu}\rangle\Big] \quad \text{(G.89)}$$

now we apply the annihilation operator $\hat{d}_{\mu''}$

$$= \quad '' \quad '' \quad '' \quad ''$$

$$\Big[g_{\mu''}(t+T)g_{\mu}(t)\langle\psi_{out}|\hat{c}^{\dagger}_{\mu'}\hat{d}^{\dagger}_{\nu'} \; 0 \quad +$$
$$+g_{\mu''}(t+T)g_{\nu}(t) \quad '' \quad '' \quad 0 \quad -$$
$$-g_{\mu''}(t+T)g_{\mu}(t) \quad '' \quad '' \quad |0\rangle\delta_{\mu''\nu} +$$
$$+g_{\mu''}(t+T)g_{\nu}(t) \quad '' \quad '' \quad |0\rangle\delta_{\mu''\mu}] \tag{G.90}$$

$$= \quad '' \quad '' \quad '' \quad ''$$

$$\Big[-g_{\nu}(t+T)g_{\mu}(t)\langle\psi_{out}|\hat{c}^{\dagger}_{\mu'}\hat{d}^{\dagger}_{\nu'}|0\rangle\delta_{\mu''\nu} +$$
$$+g_{\mu}(t+T)g_{\nu}(t) \quad '' \quad '' \quad |0\rangle\delta_{\mu''\mu}] \tag{G.91}$$

now we compute the sum over μ''

$$= \frac{1}{2}\sum_{\mu,\nu=1}^{\infty}\alpha_{\mu}\beta_{\nu}\sum_{\mu'\nu'=1}^{N} g^{*}_{\mu'}(t)g^{*}_{\nu'}(t+T)$$

$$\Big[-g_{\nu}(t+T)g_{\mu}(t)\langle\psi_{out}|\hat{c}^{\dagger}_{\mu'}\hat{d}^{\dagger}_{\nu'}|0\rangle +$$
$$+g_{\mu}(t+T)g_{\nu}(t) \quad '' \quad '' \quad |0\rangle] \tag{G.92}$$

$$= \quad '' \quad '' \quad '' \quad ''$$

$$\langle\psi_{out}|\hat{c}^{\dagger}_{\mu'}\hat{d}^{\dagger}_{\nu'} \quad [-g_{\nu}(t+T)g_{\mu}(t)\;|0\rangle + g_{\mu}(t+T)g_{\nu}(t)\;|0\rangle] \tag{G.93}$$

now we explicit the state $\langle\psi_{out}|$ (cfr. (7.34))

$$= \frac{1}{2}\,\frac{1}{2}\sum_{\mu,\nu,\mu^*,\nu^*=1}^{\infty}\alpha_{\mu}\;\beta_{\nu}\;\alpha^{*}_{\mu^*}\;\beta^{*}_{\nu^*}\sum_{\mu'\nu'=1}^{N} g^{*}_{\mu'}(t)g^{*}_{\nu'}(t+T)$$
$$[\langle 1_{c\mu^*}, 1_{c\nu^*}, 0_d| - \langle 1_{c\mu^*}, 1_{d\nu^*}| + \langle 1_{c\nu^*}, 1_{d\mu^*}| - \langle 0_c, 1_{d\mu^*}, 1_{d\nu^*}|]\;\hat{c}^{\dagger}_{\mu'}\hat{d}^{\dagger}_{\nu'}$$
$$[-g_{\nu}(t+T)g_{\mu}(t)\;|0\rangle + g_{\mu}(t+T)g_{\nu}(t)\;|0\rangle] \tag{G.94}$$

then we apply the operators on their left side, first $\hat{c}^{\dagger}_{\mu'}$

$$= \frac{1}{4}\sum_{\mu,\nu,\mu^*,\nu^*=1}^{\infty}\alpha_{\mu}\;\beta_{\nu}\;\alpha^{*}_{\mu^*}\;\beta^{*}_{\nu^*}\sum_{\mu'\nu'=1}^{N} g^{*}_{\mu'}(t)g^{*}_{\nu'}(t+T)$$
$$[\langle 0|(\delta_{\mu'\mu^*} + \delta_{\mu'\nu^*}) - \langle 1_{d\nu^*}|\delta_{\mu'\mu^*} + \langle 1_{d\mu^*}|\delta_{\mu',\nu^*} - 0]\;\hat{d}^{\dagger}_{\nu'}$$
$$[-g_{\nu}(t+T)g_{\mu}(t)\;|0\rangle + g_{\mu}(t+T)g_{\nu}(t)\;|0\rangle] \tag{G.95}$$

then we compute the sum on μ':

$$= \frac{1}{4} \sum_{\mu,\nu,\mu^*,\nu^*=1}^{\infty} \alpha_\mu \; \beta_\nu \; \alpha^*_{\mu^*} \; \beta^*_{\nu^*} \sum_{\nu'=1}^{N} g^*_{\nu'}(t+T)$$
$$\left[\langle 0|(g^*_{\mu^*}(t) + g^*_{\nu^*}(t)) - \langle 1_{\nu^*}|g^*_{\mu^*}(t) + \langle 1_{\mu^*}|g^*_{\nu^*}(t)\right] \hat{d}^\dagger_{\nu'}$$
$$[-g_\nu(t+T)g_\mu(t) \; |0\rangle + g_\mu(t+T)g_\nu(t) \; |0\rangle] \tag{G.96}$$

then we apply $\hat{d}^\dagger_{\nu'}$:

$$= \frac{1}{4} \sum_{\mu,\nu,\mu^*,\nu^*=1}^{\infty} \alpha_\mu \; \beta_\nu \; \alpha^*_{\mu^*} \; \beta^*_{\nu^*} \sum_{\nu'=1}^{N} g^*_{\nu'}(t+T)$$
$$\left[0 - \langle 0|\delta_{\nu'\nu^*} g^*_{\mu^*}(t) + \langle 0|\delta_{\nu'\mu^*} g^*_{\nu^*}(t)\right]$$
$$[-g_\nu(t+T)g_\mu(t) \; |0\rangle + g_\mu(t+T)g_\nu(t) \; |0\rangle] \tag{G.97}$$

and then compute the sum on ν':

$$= \frac{1}{4} \sum_{\mu,\nu,\mu^*,\nu^*=1}^{\infty} \alpha_\mu \; \beta_\nu \; \alpha^*_{\mu^*} \; \beta^*_{\nu^*}$$
$$\left[-\langle 0|g^*_{\nu^*}(t+T)g^*_{\mu^*}(t) + \langle 0|g^*_{\mu^*}(t+T)g^*_{\nu^*}(t)\right]$$
$$[-g_\nu(t+T)g_\mu(t) \; |0\rangle + g_\mu(t+T)g_\nu(t) \; |0\rangle] \tag{G.98}$$

then we factor the vacuum states (their scalar product is 1):

$$= \frac{1}{4} \sum_{\mu,\nu,\mu^*,\nu^*=1}^{\infty} \alpha_\mu \; \beta_\nu \; \alpha^*_{\mu^*} \; \beta^*_{\nu^*}$$
$$\left[-g^*_{\nu^*}(t+T)g^*_{\mu^*}(t) + g^*_{\mu^*}(t+T)g^*_{\nu^*}(t)\right]$$
$$[-g_\nu(t+T)g_\mu(t) + g_\mu(t+T)g_\nu(t)] \, \langle 0|0\rangle \tag{G.99}$$

G.2.12 Applying Marcin approach to HOM

$$\hat{C} \equiv \hat{c}^\dagger(t)\hat{d}^\dagger(t+T)\hat{d}(t+T)\hat{c}(t) \tag{G.100}$$
$$\hat{C} = \hat{c}^\dagger(t)\hat{c}(t)\hat{d}^\dagger(t+T)\hat{d}(t+T) \tag{G.101}$$
$$\hat{C} = \hat{n}_c(t)\hat{n}_d(t+T) \tag{G.102}$$
$$\tag{G.103}$$

so that

$$\langle\psi_{out}|C|\psi_{out}\rangle = \langle\psi_{out}|\hat{n}_c(t)\hat{n}_d(t+T)|\psi_{out}\rangle \tag{G.104}$$

where:

$$|\psi_{out}\rangle = \frac{1}{2}\sum_{\mu,\nu=1}^{\infty}\alpha_\mu\beta_\nu\left[|1_{c\mu},1_{c\nu}\rangle - |1_{c\mu},1_{d\nu}\rangle + |1_{c\nu},1_{d\mu}\rangle - |1_{d\mu},1_{d\nu}\rangle\right].$$

then we write the coincidence function and the output state in terms of time-bin operators :

$$\begin{aligned}
&\hat{c}^\dagger(t)\ \hat{c}(t)\ \hat{d}^\dagger(t+T)\ \hat{d}(t+T) = \\
&\qquad = \sum_{\lambda\varepsilon=1}^{N} g^*_\lambda(t)\ g_\lambda(t)\ g^*_\varepsilon(t+T)\ g_\varepsilon(t+T)\ \ \hat{c}^\dagger_\lambda\ \hat{c}_\lambda\ \hat{d}^\dagger_\varepsilon\ \hat{d}_\varepsilon && \text{(G.105)}\\
&\qquad = \sum_{\mu'\nu'=1}^{N} |g_\lambda(t)|^2\ |g_\varepsilon(t+T)|^2\ \ \hat{c}^\dagger_\lambda\ \hat{c}_\lambda\ \hat{d}^\dagger_\varepsilon\ \hat{d}_\varepsilon && \text{(G.106)}
\end{aligned}$$

(cfr. (G.8)), and

$$\begin{aligned}
|\psi_{out}\rangle &= \frac{1}{2}\sum_{\mu,\nu=1}^{\infty}\alpha_\mu\beta_\nu\left[\hat{c}^\dagger_\mu\hat{c}^\dagger_\nu|0\rangle - \hat{c}^\dagger_\mu\hat{d}^\dagger_\nu|0\rangle + \hat{c}^\dagger_\nu\hat{d}^\dagger_\mu|0\rangle - \hat{d}^\dagger_\mu\hat{d}^\dagger_\nu|0\rangle\right] && \text{(G.107)}\\
&= \frac{1}{2}\sum_{\mu,\nu=1}^{\infty}\alpha_\mu\beta_\nu\left[|1_{c\,\mu},1_{c\,\nu}\rangle - |1_{c\,\mu},1_{d\,\nu}\rangle + |1_{c\,\nu},1_{d\,\mu}\rangle - |1_{d\,\mu},1_{d\,\nu}\rangle\right] && \text{(G.108)}
\end{aligned}$$

(cfr. (7.14)).

This recalled, implementing these expressions we have:

$$\langle\psi_{out}|C|\psi_{out}\rangle =$$

$$= \frac{1}{4}\sum_{\mu,\nu,\mu',\nu'=1}^{\infty} \alpha^*_\mu \beta^*_\nu \alpha_{\mu'}\beta_{\nu'} \sum_{\lambda,\varepsilon=1}^{N} |g_\lambda(t)|^2 \; |g_\varepsilon(t+T)|^2$$

$$[\langle 1_{c\mu}, 1_{c\nu}| - \langle 1_{c\mu}, 1_{d\nu}| + \langle 1_{c\nu}, 1_{d\mu}| - \langle 1_{d\mu}, 1_{d\nu}|] \tag{G.109}$$

$$\hat{c}^\dagger_\lambda \, \hat{c}_\lambda \, \hat{d}^\dagger_\varepsilon \, \hat{d}_\varepsilon \, [|1_{c\mu'}, 1_{c\nu'}\rangle - |1_{c\mu'}, 1_{d\nu'}\rangle + |1_{c\nu'}, 1_{d\mu'}\rangle - |1_{d\mu'}, 1_{d\nu'}\rangle] \tag{G.110}$$

we have already seen that out of the $4 \times 4 = 16$ therms, only 4 are non zero:

term 6

$$\langle 1_{c\mu}, 1_{d\nu}|\hat{c}^\dagger_\lambda \, \hat{d}^\dagger_\varepsilon \, \hat{d}_\varepsilon \, \hat{c}_\lambda|1_{c\mu'}, 1_{d\nu'}\rangle = \tag{G.111}$$

$$= \langle 1_{d\nu}|\delta_{\mu\lambda}\hat{d}^\dagger_\varepsilon \, \hat{d}_\varepsilon \, \delta_{\lambda\,\mu'}|1_{d\nu'}\rangle \tag{G.112}$$

$$= \langle 0|\delta_{\mu\lambda} \, \delta_{\nu\,\varepsilon} \, \delta_{\varepsilon\,\nu'} \, \delta_{\lambda\,\mu'}|0\rangle \tag{G.113}$$

$$\delta_{\mu\,\lambda} \, \delta_{\nu\,\varepsilon} \, \delta_{\varepsilon\,\nu'} \, \delta_{\lambda\,\mu'} \tag{G.114}$$

term 7

$$\langle 1_{c\mu}, 1_{d\nu}|\hat{c}^\dagger_\lambda \, \hat{d}^\dagger_\varepsilon \, \hat{d}_\varepsilon \, \hat{c}_\lambda|1_{c\nu'}, 1_{d\mu'}\rangle = \tag{G.115}$$

$$\delta_{\mu\,\lambda} \, \delta_{\nu\,\varepsilon} \, \delta_{\lambda\,\nu'} \, \delta_{\varepsilon\,\mu'} \tag{G.116}$$

term 10

$$\langle 1_{c\nu}, 1_{d\mu}|\hat{c}^\dagger_\lambda \, \hat{d}^\dagger_\varepsilon \, \hat{d}_\varepsilon \, \hat{c}_\lambda|1_{c\mu'}, 1_{d\nu'}\rangle = \tag{G.117}$$

$$\delta_{\nu\,\lambda} \, \delta_{\mu\,\varepsilon} \, \delta_{\lambda\,\mu'} \, \delta_{\varepsilon\,\nu'} \tag{G.118}$$

term 11

$$\langle 1_{c\mu}, 1_{d\nu}|\hat{c}^\dagger_\lambda \, \hat{d}^\dagger_\varepsilon \, \hat{d}_\varepsilon \, \hat{c}_\lambda|1_{c\mu'}, 1_{d\nu'}\rangle = \tag{G.119}$$

$$\delta_{\mu\,\lambda} \, \delta_{\nu\,\varepsilon} \, \delta_{\lambda\,\mu'} \, \delta_{\varepsilon\,\nu'} \tag{G.120}$$

so, summarizing and taking into account the signs, we have:

$$\begin{aligned}
&\langle\psi_{out}|C|\psi_{out}\rangle = \\
&= \frac{1}{4}\sum_{\mu,\nu,\mu',\nu'=1}^{\infty} \alpha^*_\mu \beta^*_\nu \alpha_{\mu'} \beta_{\nu'} \sum_{\lambda,\varepsilon=1}^{N} |g_\lambda(t)|^2 \; |g_\varepsilon(t+T)|^2 \\
&[\delta_{\mu\,\lambda}\,\delta_{\nu\,\varepsilon}\,\delta_{\varepsilon\,\nu'}\,\delta_{\lambda\,\mu'} - \delta_{\mu\,\lambda}\,\delta_{\nu\,\varepsilon}\,\delta_{\lambda\,\nu'}\,\delta_{\varepsilon\,\mu'} - \delta_{\nu\,\lambda}\,\delta_{\mu\,\varepsilon}\,\delta_{\lambda\,\mu'}\,\delta_{\varepsilon\,\nu'} + \\
&\qquad + \delta_{\mu\,\lambda}\,\delta_{\nu\,\varepsilon}\,\delta_{\lambda\,\mu'}\,\delta_{\varepsilon\,\nu'}]
\end{aligned} \tag{G.121}$$

summing over λ and ε and using $\sum_\lambda \delta_{\mu\,\lambda}\delta_{\lambda\,\mu'} = \delta_{\mu\,\mu'}$

$$\begin{aligned}
&= \left(\sum_{\lambda,\varepsilon=1}^{N} |g_\lambda(t)|^2 \; |g_\varepsilon(t+T)|^2\right) \frac{1}{4}\sum_{\mu,\nu,\mu',\nu'=1}^{\infty} \alpha^*_\mu \beta^*_\nu \alpha_{\mu'} \beta_{\nu'} \\
&\qquad [\delta_{\mu\,\mu'}\,\delta_{\nu\,\nu'} - \delta_{\mu\,\nu'}\,\delta_{\nu\,\mu'} - \delta_{\nu\,\mu'}\,\delta_{\mu\,\nu'} + \delta_{\mu\,\mu'}\,\delta_{\nu\,\nu'}]
\end{aligned} \tag{G.122}$$

summing over μ' and ν'

$$\begin{aligned}
&= \left(\sum_{\lambda,\varepsilon=1}^{N} |g_\lambda(t)|^2 \; |g_\varepsilon(t+T)|^2\right) \\
&\quad \left(\frac{1}{4}\sum_{\mu,\nu=1}^{\infty} \alpha^*_\mu \beta^*_\nu \alpha_\mu \beta_\nu - \frac{1}{4}\sum_{\mu,\nu=1}^{\infty} \alpha^*_\mu \beta^*_\nu \alpha_\nu \beta_\mu - \right. \\
&\qquad \left. -\frac{1}{4}\sum_{\mu,\nu=1}^{\infty} \alpha^*_\mu \beta^*_\nu \alpha_\nu \beta_\mu + \frac{1}{4}\sum_{\mu,\nu=1}^{\infty} \alpha^*_\mu \beta^*_\nu \alpha_\mu \beta_\nu\right)
\end{aligned} \tag{G.123}$$

$$\begin{aligned}
&= \left(\sum_{\lambda,\varepsilon=1}^{N} |g_\lambda(t)|^2 \; |g_\varepsilon(t+T)|^2\right) \\
&\qquad \left(\frac{1}{2}\sum_{\mu,\nu=1}^{\infty} |\alpha_\mu|^2\,|\beta_\nu|^2 - \frac{1}{2}\sum_{\mu,\nu=1}^{\infty} \alpha^*_\mu \beta^*_\nu \alpha_\nu \beta_\mu\right)
\end{aligned} \tag{G.124}$$

and finally

$$\langle\psi_{out}|C|\psi_{out}\rangle =$$
$$= \left(\sum_{\lambda,\varepsilon=1}^{N} |g_\lambda(t)|^2 \; |g_\varepsilon(t+T)|^2\right)\left(\frac{1}{2} - \frac{1}{2}\sum_{\mu,\nu=1}^{\infty} \alpha_\mu^* \beta_\nu^* \alpha_\nu \beta_\mu\right) \qquad (G.125)$$

where we have used the normalization condition:

$$\sum_{\mu=1}^{\infty} |\alpha_\mu|^2 = \sum_{\nu=1}^{\infty} |\beta_\nu|^2 = 1$$

G.3 Recalc with Heisemberg formalism

(this section follows the approach in [HOM87])

Here we want to use another (equivalent) strategy to compute the HOM coincidence function, using the Heisenberg approach. This means that we define the coincidence operator $\hat{C}$ as in (**??**):

$$\hat{C} = \hat{a}^\dagger(t)\,\hat{b}^\dagger(t+T)\,\hat{b}(t+T)\,\hat{a}(t) \qquad (G.126)$$

$$|\psi_{in}\rangle = |1_a, 1_b\rangle \qquad (G.127)$$

and then, instead of calculating the expectation value of this operator on the *output state* of the beam splitter $|\psi_{out}\rangle$, we apply the unitary "beam splitter transformation" operator $\hat{U}$ to the operator, and compute the expectation value on the *input state* $|\psi_{in}\rangle$, where

$$\hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \qquad (G.128)$$

To transform an operator we have to write

$$\hat{C}_{out} = \hat{U}^\dagger \hat{C} \hat{U} \qquad (G.129)$$

but instead of this "formal" procedure, we just transform the single creation/annihilation operators, knowing that:

$$\begin{cases} \hat{a}(t) = \frac{\hat{c}^\dagger(t-t_1)+i\hat{d}^\dagger(t-t_1+T)}{\sqrt{2}} \\ \hat{b}(t) = \frac{i\hat{c}^\dagger(t-t_1+T)+\hat{d}^\dagger(t-t_1)}{\sqrt{2}} \end{cases} \tag{G.130}$$

$$P = \langle\psi_{in}|\hat{U}^\dagger\hat{C}\hat{U}|\psi_{in}\rangle \tag{G.131}$$

$$= \langle\psi_{in}|\hat{U}^\dagger \left[\hat{a}^\dagger(t)\,\hat{b}^\dagger(t+T)\,\hat{b}(t+T)\,\hat{a}(t)\right]\hat{U}|\psi_{in}\rangle \tag{G.132}$$

$$= \langle\psi_{in}|\left[\frac{\hat{c}^\dagger(t-t_1)+i\hat{d}^\dagger(t-t_1+T)}{\sqrt{2}}\;\frac{i\hat{c}^\dagger(t-t_1+T)+\hat{d}^\dagger(t-t_1)}{\sqrt{2}}\right] \tag{G.133}$$

$$\left[\frac{\hat{c}^\dagger(t-t_1+T)+i\hat{d}^\dagger(t-t_1+T)}{\sqrt{2}}\;\frac{i\hat{c}^\dagger(t-t_1+T)+\hat{d}^\dagger(t-t_1)}{\sqrt{2}}\right]|\psi_{in}\rangle \tag{G.134}$$

$$\left[\hat{b}(t+T)\,\hat{a}(t)\right]|\psi_{in}\rangle \tag{G.135}$$

$$P(T) = \tag{G.136}$$

Appendix H

Other appendices

H.1 Trasformata e delta

$$\delta(k - k') = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{-i\,(k-k')\,x}\, dx \tag{H.1}$$

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