An introduction to Information Theory and some of its applications: Black Hole Information Paradox and Renormalization Group Information Flow

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Classical Information Theory

Random variable

$X \equiv (x, \mathcal{A}_X, \mathcal{P}_X)$

x (variable)

$\mathcal{A}_X \equiv \{a_1, a_2, \ldots, a_n\}$ (possible outcomes)

$\mathcal{P}_X \equiv \{p_1, p_2, \ldots, p_n\}$ (probabilities)

Information content
(single outcome)

$$h(x = a_i) \equiv \log \frac{1}{p_i}$$

Entropy
(info. content of whole variable)

$$H(X) \equiv \sum_i p_i \log \frac{1}{p_i}$$

Shannon's Source coding theorem

N i.i.d. random variables each with entropy H(X) can be *compressed* into more than NH(X) bits with negligible risk of information loss, as $N \rightarrow \infty$;
conversely if they are compressed into fewer than NH(X) bits is is virtually certain that information will be lost. (Mackay - Information Theory, Inference, and Learning Algorithms (2003))

Example
(tomorrow's weather)

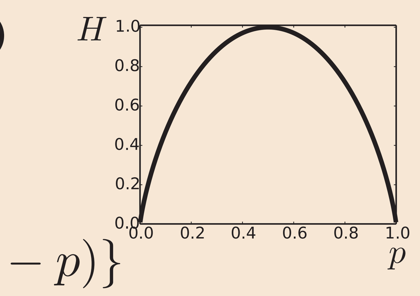
 vs 

$\mathcal{A}_X \equiv \{☀, ☂\}$ (binary variable)

$\mathcal{P} = \{p(☀) = 1, \quad p(☂) = 0\}$ (not informative)

$\mathcal{P} = \{p(☀) = 0.5, \quad p(☂) = 0.5\}$ (informative)

Entropy (binary variable)



$\mathcal{A}_X = \{0, 1\}$

$\mathcal{P}_X = \{p_0 = p, \; p_1 = (1-p)\}$

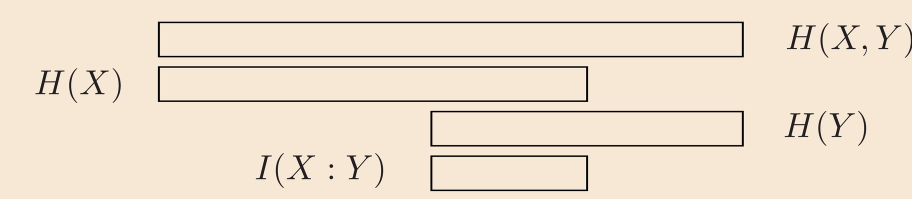
$$H(X) = p \log_2 \frac{1}{p} + (1-p) \log_2 \frac{1}{(1-p)}$$

Other definitions

$p(x, y)$ joint probability

$$H(X,Y) \equiv \sum_{x,y} p(x,y) \log \frac{1}{p(x,y)}$$ joint entropy

$$I(X:Y) \equiv H(X) + H(Y) - H(X,Y)$$ mutual information



Insights

- information content is function of the probability only:
  $h(x) = h[p(x)]$
- joint probability of independent variables is the product:
  $p(x,y) = p(x)\, p(y)$
- intuition: information content is additive:
  $h(x,y) = h(x) + h(y)$

$$\Rightarrow \quad h(x) = \log[f(p)]$$

- information content maximal for equiprobable distribution (see plot of binary entropy in the example on the left)

$$\Downarrow$$

$$h(x = a_i) = \log \frac{1}{p_i}$$

Quantum Information Theory

Hilbert space

$\mathcal{H} = \operatorname{span}\{|1\rangle, \ldots, |n\rangle, \ldots\}$

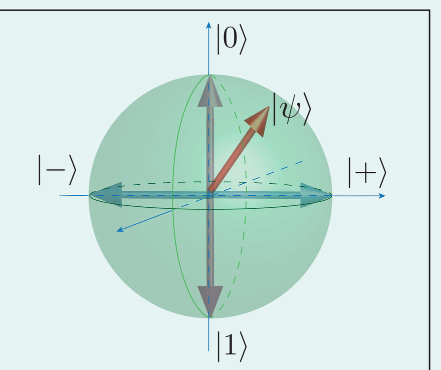
$|\psi\rangle = c_1|1\rangle + \ldots + c_n|n\rangle + \ldots$

$\forall i, \; c_i \in \mathbb{C}$

Qubit

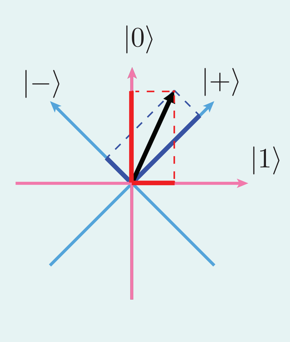
$\mathcal{H} = \operatorname{span}\{|0\rangle, |1\rangle\}$

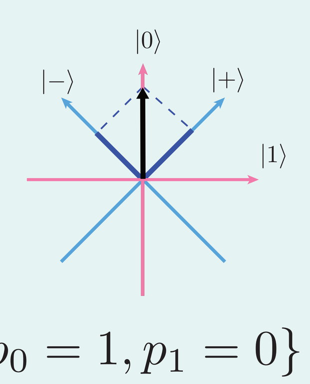
$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$



Quantum complementarity

$$\begin{cases} |+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \\ |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \end{cases}$$





POVMs $\{\hat{P}_0, \hat{P}_1\}$; $\{\hat{P}_+, \hat{P}_-\}$

probabilities $\{p_0 = 1, p_1 = 0\}$ $\{p_+ = 0.5, p_- = 0.5\}$

Pure states & mixed states

$$\rho_0 \equiv |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\rho_\psi \equiv |\psi\rangle\langle\psi| = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} (c_0^*, c_1^*) = \begin{pmatrix} c_0 c_0^* & c_0 c_1^* \\ c_1 c_0^* & c_1 c_1^* \end{pmatrix}$$

pure states

$\{|\psi_1\rangle, \ldots, |\psi_n\rangle\}$ possible states ; $\{p_1, \ldots, p_n\}$ probabilities

$$\rho = \sum_i p_i \, \rho_i = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$$ mixed state

Holevo theorem

Suppose Alice prepares a state ρ_X where $X = \{|x_1\rangle, \ldots, |x_n\rangle\}$ with probabilities $\{p_1, \ldots, p_n\}$.
Bob performs a measurement described by POVM $\{\hat{P}_1, \ldots, \hat{P}_n\}$ on that state, with measurement outcome Y.
For any such measurement Bob may do:

$$I(X:Y) \leq S(\rho) - \sum_x p_x S(\rho_x) \; \Big(\leq H(X)\Big)$$

where $\rho = \sum_x p_x \rho_x$. The equality holds in the case the $\{|x_1\rangle, \ldots, |x_n\rangle\}$ are all orthogonal.

(Nielsen, Chuang - Quantum Computation and Quantum Information (2001))

No-cloning theorem

Given two states, non parallel (distinct) and non orthogonal (not perfectly distinguishable), if we are given one of the two, *but we don't know which one*, then there is no unitary operation which starts with two systems (the "original" and the "copy"), the first in the unknown state and the second in an initial known state, and ends with both in the unknown state

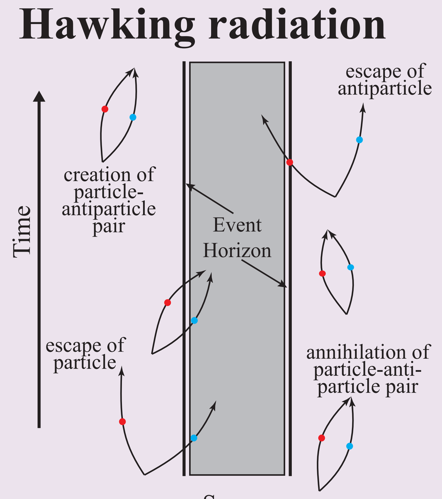
$\{|\psi_1\rangle, |\psi_2\rangle\}$

$0 < |\langle\psi_1|\psi_2\rangle| < 1$

$$\cancel{\hat{U}(\underbrace{|\psi_i\rangle}_{\text{orig.}}\underbrace{|0\rangle}_{\text{copy}}) = \underbrace{|\psi_i\rangle}_{\text{orig.}}\underbrace{|\psi_i\rangle}_{\text{copy}}}$$

Black Hole Information Paradox

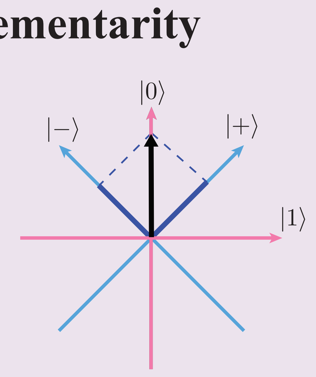
Hawking radiation



- due to vacuum fluctuations couples of particle-antiparticle are continuously created and annihilate
- if one of the two particles falls inside the event horizon, the other escapes
- the Hawking radiation decreases the energy of the BH, and eventually leads to its *complete evaporation*
- the Hawking is emitted in a *totally mixed state*

Black Hole complementarity

- a possible solution to the cloning paradox is the idea that the two copies of the information are actually encoded in two complementary bases.
- So, although there exist two copies of the same information, they are never accessible at the same time, or for the same observer

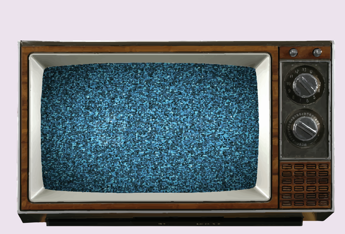


The paradox

- without loss of generality, we can assume that the initial state of the BH is a *pure state*
- at the end of the evaporation, only the Hawking radiation remains, which is in a *mixed state*
- Quantum Mechanics describes the evolution of isolated systems as the action of an *unitary operator*
- Since no unitary operator can transform a pure state into a mixed state, this leads to a *paradox*

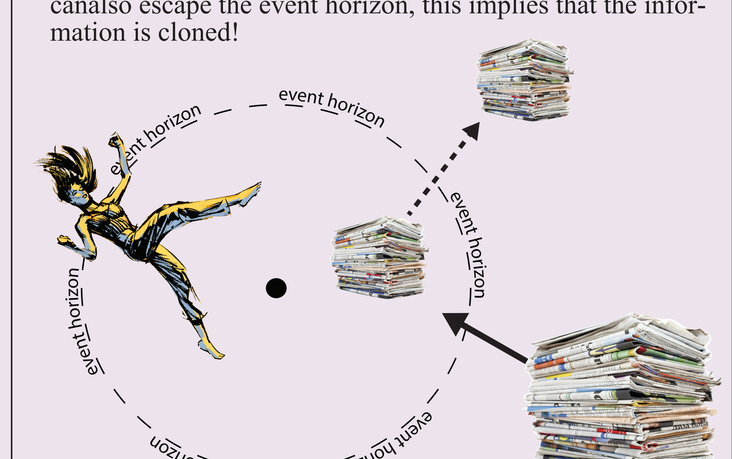
$$\cancel{\hat{U}|\psi\rangle = \rho}$$

- it is possible to re-formulate the paradox in terms of information: the totally mixed final state is the quantum analog of the classical equiprobable distribution. This state shares no information with the initial state: the outcome of an observation of the final state has no mutual information with the initial state. Where is the *missing information*?



Cloning?

- if we assume that somehow the initial information escapes the BH, this implies the breaking of the no-cloning theorem
- indeed, from the point of view of the in-falling observer, crossing the event horizon has no physical effects. So, it is safe to assume that after the crossing the information is intact.
- if on the other hand we assume that the same information canalso escape the event horizon, this implies that the information is cloned!



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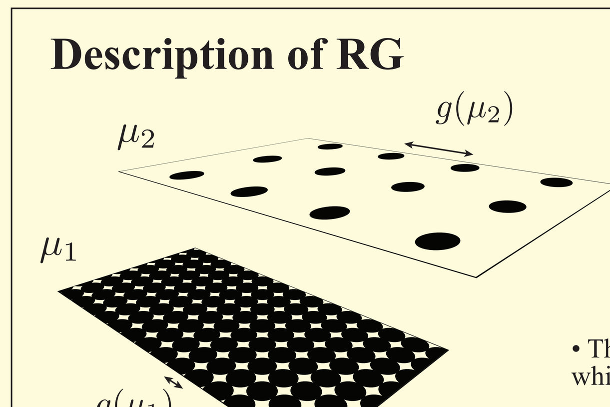
Renormalization Group Information Flux

Description of RG



- The dynamics of a composite system can be described by the interactions between its components. At a certain scale (graining) μ_1 the elementary components are defined, and the interaction constant g between them. At an higher scale (more coarse graining) μ_2, the elementary components can be "clusters" (blocs) of elementary components at the finer scale. At the higher scale the interacting constant is in principle changed. So in general we have an interaction constant which is function of the scale: $g(\mu_i)$.
- The main idea of the Renormalization Group (RG) is to define an operator $\hat{G}$ which applied to the interaction constant at the smaller scale:

$$g(\mu_2) = \hat{G} \; g(\mu_1) \quad \text{(group operator action)}$$

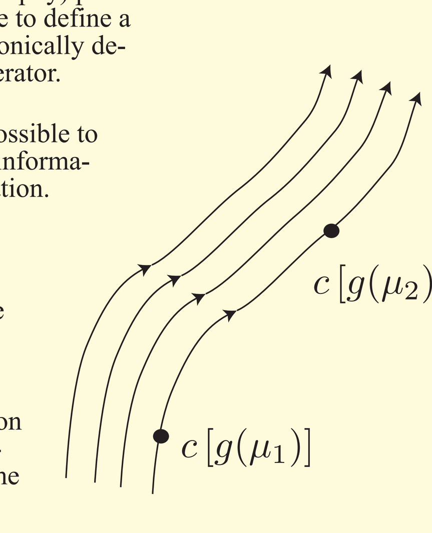
- The relationship between the group operator $\hat{G}$ and the interaction constant is described by the Callan-Symanzik equation, which enforces the consistency between the descriptions at different scales:

$$\left[m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} + n\gamma\right] C^{(n)}(x_1, \ldots, x_n; m, g) = 0$$ Callan-Symanzik equation

where:
m is the mass
C is the correlation function between the $(x_1, \ldots, x_n)$ elements of the system
β, γ are two functions that "compensate" the effect of the scale change, in order for the description (i.e. the correlation function) at the different scales to be consistent. In particular, β captures the change the change of the cou- pling constant, while γ captures the change of the field itself.

Information flow

- In 1986 Zamolodchikov (see bibliography) proves that (in a 2D case) it is always possible to define a function, the c-function, which monotonically decreases under the action of the RG operator.
- Because of this monotonicity, it is possible to give to this function the meaning of "information flow" along the group transformation.
- This also gives rise to irreversibility under the group transformations: the group is not formally a group because there for each operator there is no inverse.
- This irreversibility captures the notion that despite the consistency, in the description at a coarse grained scale some information about the description at a finer scale has been lost.
- What we have seen shows how Information Theory is an ideal tool to use for future research within the framework of RG.



$$c[g(\mu_1)] \leq c[g(\mu_2)]$$

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