Notes on Calculus

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Contents

Chapter 1

Introduction

In this chapter we will introduce the main concepts.

1.1 Functions

A function can be imagined as a 'processing device', which processes numbers.
In the most simple case, there is a single number as *input*, and a single number as *output*. In other words, a function can be defined as a relationships between two (or more) numbers that are defined by the properties of the function. The *input* (variable) is changeable, and the *output* depends on the properties of the function ('commands' included to the 'processing device' that tell it how to work with the *input*).

1.2 Limits

In this section we will introduce the definition and concept of limits.

1.2.1 Zeno paradox

Zeno paradox tells us a story of how correct understanding of concepts of the limit, infinity and infinitesimality can explain the things that seems impossible. The story starts with the competition between the fastest runner and the turtle to run to the finish first. As the turtle is obviously slower it has an advantage and starts closer to the finish. The star gun shots, and competition begins. The champions start the moving, and after some time they change their positions closer to the finish, and, obviously, the runner goes the larger distance, but the distance between the runner and the turtle is not 0 because in this time the turtle also moved from the starting point to another one. After the second perion of time the runner and the turtle also moved from their points further, and the distance between them became smaller but still not zero. This steps could be repeated as many times as you want up to infinite, and the distance between the champions and the turtle will never be zero, because turtle is continuing moving while the runner moves. So, according to the paradox, the runner will never catch up the turtle. But intuitively we understand that it is not true. But what makes the contradiction?

1.2.2 Velocity and speed

We know that the moving objects has the velocity that is the distance the object goes through the time of moving.

$$v = \delta S/\delta t \tag{1.1}$$

With this formula it looks like that the paradox is quite acceptable. But in this way we can calculate the overage velocity in the selected time periods and selected space intervals. The absolute speed - the velocity of the object at the defined time point at the defined point of space - that is the characteristic of the object which is interesting for us. Itself the 'absolute

speed' by the definition is oxymoron as the object at a certain point has no speed because it does not move in the space!
Here we go to the *limits*.

1.2.3 Limits definition

For the more interesting functions, limits are the points(numbers) that the function never achieve. But usually the limit of the function is the point where the function tends to achieve the value (and it can be defined in the point as well as not) but does not cross it. For instance, limit of x2 with x -> 0 is equal 0 and at 0 the function is defined but does not cross the limit point.
Also, limits could be *right* (for the larger values of the variable) and *left* (for the lower values of the variable).

1.3 Derivatives

[...]

Chapter 2

Differential Equations

2.1 Vector Spaces

2.1.1 definition of a vector space

We briefly remind the definition of *vector space*.
To "build" a vector space we need:

- a set of elements: S (called "vectors")
- a "function" called "sum", defined on S, such that to any pair of elements in S it associates an element in S:

$$\forall \vec{u}, \vec{v} \in S \rightarrow (\vec{u} + \vec{v}) \in S \tag{2.1}$$

- a function called "multiplication with a number" (or "multiplication with a scalar") such that to any pair of an element $\vec{v}$ in S and a real number α in $\mathbb{R}$, it associates an element of S:

$$\vec{v} \in S, \alpha \in \mathbb{R} \rightarrow \alpha\vec{v} \in S \tag{2.2}$$

The two functions need the following properties:

properties of the *sum*

- commutative:

$$\vec{v} + \vec{u} = \vec{u} + \vec{v}, \qquad \forall \vec{v}, \vec{u} \in S \tag{2.3}$$

- associative:

$$(\vec{v} + \vec{u}) + \vec{w} = \vec{v} + (\vec{u} + \vec{w}), \qquad \forall \vec{v}, \vec{u}, \vec{w} \in S \tag{2.4}$$

- with a neutral element $\vec{0}$ (called "null vector"), such that:

$$\vec{v} + \vec{0} = \vec{v}, \qquad \forall \vec{v} \in S \tag{2.5}$$

- with a symmetric element $-\vec{v}$:

$$\forall \vec{v} \in S \exists (-\vec{v}) \in S : \vec{v} + (-\vec{v}) = \vec{0}. \tag{2.6}$$

properties of the *multiplication with a scalar*

- $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$ (2.7)

- $1\vec{v} = \vec{v}$ (2.8)

- $\alpha(\vec{v} + \vec{u}) = \alpha\vec{v} + \alpha\vec{u}$ (2.9)

- $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$ (2.10)

we notice that from these properties we have:

$$\alpha\vec{v} = \vec{0} \qquad \text{iif} \qquad \alpha = 0 or \vec{v} = \vec{0} \tag{2.11}$$

which is called “law of the cancellation of the product”. Here and in the following “iif” stands for “if, and only if”.

2.1.2 Subspaces

A subset S_0 of S is defined a “subspace of S” when:

$$\vec{v}, \vec{u} \in S_0 \quad \Rightarrow \quad \vec{v} + \vec{u} \in S_0 \tag{2.12a}$$

$$\vec{v} \in S_0, \alpha \in \mathbb{R} \quad \Rightarrow \quad \alpha\vec{v} \in S_0 \tag{2.12b}$$

As a consequence of this definition, any subspace S_0 of S includes the null vector $\vec{0}$ of S, and it also holds that $\vec{v} \in S_0 \;\Rightarrow\; -\vec{v} \in S_0$.
If we consider the two operation of *sum* and *multiplication with a scalar* defined on S, restricted only to the elements of S_0, then also S_0 is a vector space.

2.1.3 Linear combinations

Given a set of k vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ of S, and set of k real numbers $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, we can compute the vector:

$$\alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \ldots + \alpha_k\vec{v_k} \tag{2.13}$$

which will be called “*linear combination* of the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ with *coefficients* $\alpha_1, \alpha_2, \ldots, \alpha_k$”.

Linear dependence, linear independence

If all the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k$ are equal to zero, the linear combination (2.13) is equal to the null vector. However, a linear combination can be equal to the null vector also in a case when not all the coefficients are zero. So, this leads to the two following definitions.
Given a set of vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ of S,

linear independence

if the only linear combination of those vectors that results in the null vector is the one with all zero coefficients, the set of vectors is said *linearly independent*;

linear dependence

if there exist one or more linear combinations of those vectors, that results in the null vector, and where not all the coefficients are zero, the set of vectors is said *linearly dependent*.

Basis of a vector space

If we have n linearly independent vectors

$$\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\} \tag{2.14}$$

and if it happens that any vector of S can be written as a linear combination of the basis vectors (2.14), i.e.

$$\forall \vec{v} \in S\ \exists\{\alpha_1, \ldots, \alpha_n\} \in \mathbb{R} : \quad \vec{v} = \alpha_1 \vec{e}_1 + \ldots + \alpha_n \vec{e}_n \tag{2.15}$$

then the set of vectors (2.14) is called a basis of the vector space.
It can be proven that if there exist a basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$ of S with n elements, then any other basis of S will have n elements. Then, the number n is called *the dimension* of the vector space.

2.1.4 Important remark on the vector spaces

It is important to notice that in the definition of *vector space* there is no description of the elements of it (the vectors). The definition only describes the relations between the elements (the sum, the multiplication with a scalar, etc.). So, as long as we have the two operations (sum and product with scalar), with the needed properties, we can define a vector space.
In particular, we can define a vector space where the elements are *functions*. The sum of two functions can be defined as sum of the two values each function assumes on each value of the independent variable:

$$f : x \in \mathbb{R} \quad \to \quad f(x) \in \mathbb{R} \tag{2.16a}$$
$$g : x \in \mathbb{R} \quad \to \quad g(x) \in \mathbb{R} \tag{2.16b}$$
$$f + g : x \in \mathbb{R} \quad \to \quad f(x) + g(x) \in \mathbb{R} \tag{2.16c}$$

and similarly for the multiplication with a scalar:

$$\alpha f : x \in \mathbb{R} \ \to \ \alpha f(x) \in \mathbb{R} \tag{2.17}$$

2.2 Differential equations

2.2.1 Classification of differential equations

In general, a differential equation is an equation that contains an unknown function $f(x)$ and its derivatives. A differential equation of order n is an equation that includes the independent variable t, an unknown function $f(t)$, and its derivatives up to derivative of order n.

Linearity

We can use a compact notation considering a "function of functions" F (sometimes this is called an *operator*). E.g., if we consider $F[f] = \sin f + af^2 - bf'$, where we consider the sine of the unknown function, the square of the unknown function, and its second derivative.
We can imagine an *analogy* between

- the way in which a *function* f takes a variable x and "combines" it in some way, maybe creating a polynomial:

$$f \ : \ x \quad \to \quad ax^2 + bx + c \tag{2.18}$$

and

- the way in which an *operator* F takes a function f (and its derivatives) and "combines" it in some way, maybe following the analogy of the polynomial:

$$F \ : \ f \quad \to \quad af^2 + bf + cf' + d \tag{2.19}$$

(with respect to (2.18), here we have also added a term with the first derivative).
So, we can "import" the concepts of *linear* (only first power), *polynomial* (any power), and "non-algebraic" (also called transcendental, where we use trigonometric, exponential, logarithmic terms, etc.), and apply those concepts also to the operators, and to the differential equations that we can create with the operators.

Constant or non-constant coefficients

Following the analogy of the previous paragraph, in the case of linear or polynomial differential equations, we have the coefficients of the polynomial. As an example, in (2.18) and (2.19) we have the coefficients a, b, c (and d). Now, those coefficients may be constant, or they may be functions of the independent variable of the unknown function. In the first case we call the differential equation "with constant coefficients":

$$af^2(x) + bf(x) + cf'(x) + d = 0 \tag{2.20}$$

and in the case they also depend on x we call the differential equation "with non-constant coefficients":

$$a(x)f^2(x) + b(x)f(x) + c(x)f'(x) + d(x) = 0. \tag{2.21}$$

Note:
We should check in the text accompanying the equation, or check in the context, to understand whether the coefficients are constant, or they depend on the independent variable of the unknown function. It is not enough if the equation is written without the explicit dependence of the coefficients.

Partial derivatives

The unknown function f can be a function of more than one variable, as e.g. $f(x, y) = ax^2 + by^2$. For functions of more than one variable, it is possible to define *partial derivatives*, i.e. derivative that considers all the independent variables as constants, except for one, that is considered as the only variable:

$$\begin{aligned} \frac{\partial}{\partial x} f(x, y) &= \frac{\partial}{\partial x}\left(ax^2 + by^2\right) \\ &= 2ax + by^2 \end{aligned} \tag{2.22}$$

Notice how in the Leibnitz notation, the symbol d is replaced with the symbol ∂, as in $\frac{\partial}{\partial x}$.

Summary

In summary, we can classify differential equations with respect to several things.

- Wether the differential equation contains partial derivatives or not:
  - if the differential equation does not contain partial derivatives, it is called **"ordinary"**

    $$\frac{d}{dx} f(x) + 2f(x) = 0 \tag{2.23}$$

  - if the differential equation contains partial derivatives, it is called **"ordinary, with partial derivatives"**

    $$\frac{\partial}{\partial x} f(x, y) + 2\frac{\partial}{\partial y} f(x, y) = 0 \tag{2.24}$$

- Wether the unknown function appears only in polynomials, with the first power:
  - if the unknown function appears only in polynomials, with the first power, we have a **linear** differential equation

    $$a\frac{d^2}{dx^2} f(x) + b\frac{d}{dx} f(x) + cf(x) = g(x). \tag{2.25}$$

  - if the unknown function appear with a power higher than 1, or it appears as the argument of a *non algebraic* (transcendental) function the differential equation is **non linear**, e.g.

$$a\frac{d^2}{dx^2}f(x) + b\frac{d}{dx}f(x) + cf^2(x) + df(x) = 0 \tag{2.26}$$

- Whether the coefficients are constant or not:
  - if the coefficients of a polynomial (linear) differential equation are constant we call it "(differential equation) with constant coefficients";
  - if the coefficients of a polynomial (linear) differential equation are a function of the independent variable (the usual names for this variable are x, or t) we call it "(differential equation) with non-constant coefficients".
- Whether the equation has or not a term without the unknown function:
  - if the term without the unknown function is zero, the equation is called **homogeneous**:

$$\frac{d^2}{dx^2}f(x) + a\frac{d}{dx}f(x) + bf(x) = 0 \tag{2.27}$$

  - if the term without the unknown function is non-zero, the equation is called **non-homogeneous**:

$$\frac{d^2}{dx^2}f(x) + a\frac{d}{dx}f(x) + bf(x) = c \tag{2.28}$$

- The **order** of the differential equation is the highest derivative of the unknown function that appears in the equation.

2.2.2 simple forms

The most simple differential equation is the following:

$$\frac{d}{dx}f = g \tag{2.29}$$

where the function $g(x)$ is some function of the independent variable x. This differential equation is solved *integrating* the function g:

2.2.3 Superposition principle

Let's consider a linear differential equation of order n:

$$a_0 f + a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_n f^{(n)} = 0. \tag{2.30}$$

If it helps you, you can consider a compact notation, use the symbol $F_l[]$ (where the footer "l" stands for "linear") and imagine that:

$$a_0 f + a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_n f^{(n)} = F_l[f] \tag{2.31}$$

This symbol represents an operator, that takes a function f, and returns the expression $a_0 f + a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_n f^{(n)}$ (which formally is another function):

$$F_l \ : \ f \quad \rightarrow \quad a_0 f + a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_n f^{(n)}. \tag{2.32}$$

Now, the key observation is that if we have a function f that satisfies the equation (2.30), and we consider a real number α, then *also the function* (αf) *will satisfy the equation*

$$\begin{aligned} &a_0 f + a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_n f^{(n)} = 0 \ \Rightarrow \\ &\quad a_0 \alpha f + a_1 \alpha f^{(1)} + a_2 \alpha f^{(2)} + \ldots + a_n \alpha f^{(n)} = 0 \end{aligned} \tag{2.33}$$

which in the compact notation can be written as

$$F_l[f] = 0 \;\Rightarrow\; F_l[\alpha f] = 0 \tag{2.34}$$

Similarly, if we consider two functions f and g which satisfy (2.30):

$$\begin{aligned} &F_l[f] = 0 \\ &F_l[g] = 0 \end{aligned} \tag{2.35}$$

then also the *sum function* $(f + g)$ satisfies equation (2.30):

$$\begin{aligned} a_0(f+g) + a_1(f^{(1)} + g^{(1)}) + a_2(f^{(2)} + g^{(2)}) + \ldots \\ + a_n(f^{(n)} + g^{(n)}) = 0. \end{aligned} \tag{2.36}$$

In summary, given two solutions f and g of the linear differential equation (2.30), and two real numbers α and β, we have that also the function $(\alpha f + \beta g)$ is a solution of the differential equation (2.30). In figure 2.1 we have a plot of two functions f, and g and their linear combination $(\alpha f + \beta g)$.

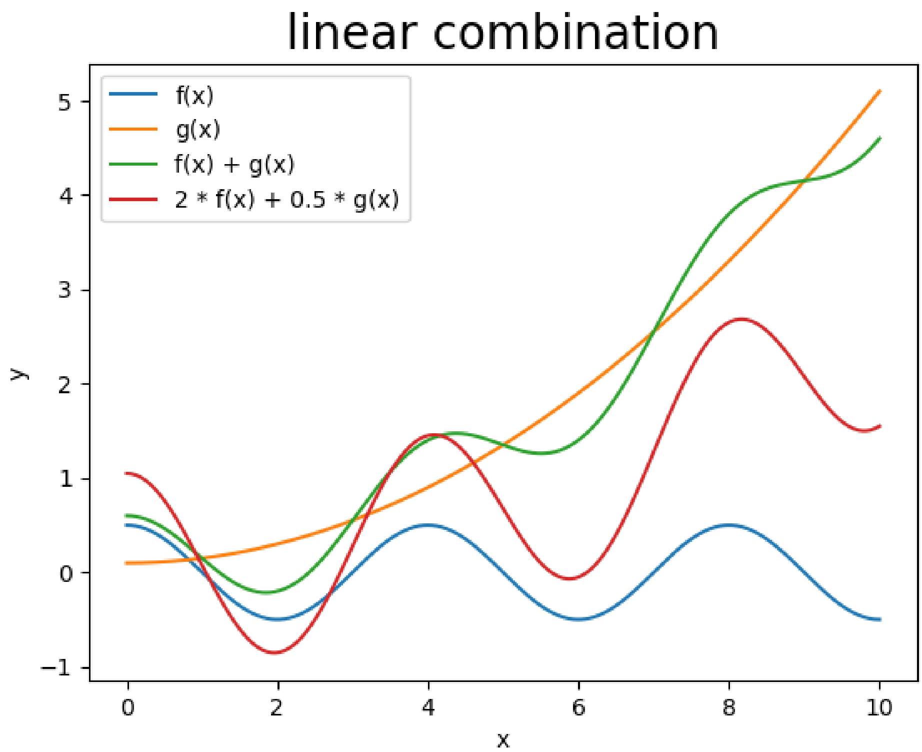

Figure 2.1: A plot of two functions and their linear combinations. In the last linear combination (red line) we can see how the oscillating shape is enhanced (higher coefficient in the linear combination) and the rising behaviour is reduced (lower coefficient in the linear combination).

It is worth to notice that if the operator F_l was not linear, i.e. if the equation was not a linear equation, this was not true. As an example, if in the differential equation the unknown function appears with a power of 2:

$$af^2 - bf' = 0 \tag{2.37}$$

and if $f(x)$ and $g(x)$ are solutions, $[f(x) + g(x)]$ are not necessarily a solution, because the square of a sum gives an extra term, and the resulting expression:

$$\begin{aligned} &a(f+g)^2 - b(f' + g') = \\ &af^2 + ag^2 + 2afg - bf' - bg' \end{aligned} \tag{2.38}$$

may or may not be equal to zero.

2.2.4 Higher order linear homogeneous differential equations

(see, [Pet66, chap 4, section 27, pag 89] [Giu83, oss. 17.1, pag 188])
If we consider an homogeneous linear differential equation of order $n > 1$:

$$a_0 f + a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_n f^{(n)} = 0 \tag{2.39}$$

it is always possible to transform it into a system of first order differential equations.
It is first convenient to rewrite the (2.39) as:

$$f^{(n)} = F_l[x, f, f^{(1)}, f^{(2)}, \ldots, f^{(n-1)}] \tag{2.40}$$

Where we have isolated the highest order derivative on the left, and we have used a compact notation, defining:

$$\begin{aligned} &F_l[x, f, f^{(1)}, f^{(2)}, \ldots, f^{(n-1)}] = \\ &\quad - a_0 f - a_1 f^{(1)} - a_2 f^{(2)} - \ldots - a_{n-1} f^{(n-1)} \end{aligned} \tag{2.41}$$

Then we introduce new functions, as $f_1 = f$, then $f_2 = f'$, and then for the derivatives higher than the first, as $f_3 = f_2' = f''$, $f_4 = f_3'$, etc.. Then putting everything together in a system of differential equations, we have:

$$\begin{cases} f_1' = f_2 \\ f_2' = f_3 \\ \quad \ldots \\ f_n' = F_l[x, f_1, f_2, f_3, \ldots, f_n] \end{cases} \tag{2.42}$$

where as desired, only first order differential equations appear.

2.3 The Cauchy problem

If we consider the simple equation (2.29), we have seen that the solution is found just integrating both sides of the equation:

$$\begin{aligned} &f'(x) = g(x) \\ &\int f'(x)\mathrm{d}x = \int g(x)\mathrm{d}x \\ &f(x) = i(x) + c \end{aligned} \tag{2.43}$$

where we have called $i(x)$ the indefinite integral of $g(x)$. This shows us that a first order differential equation doesn't have a single solution, but a "family" of solutions, i.e. a function plus an unknown constant. It can be shown that for a differential equation of order n, the number of unknown constants is equal to the order n. This can be understood intuitively, thinking that we need to integrate n times to go from $f^{(n)}(x)$ to $f(x)$.
So, if we want to to ask for *one single solution*, i.e. one specific function as solution, we can not give just a differential equation, but we need to ask for additional *conditions*. And we need to ask for a number of conditions sufficient to fix all the unknown constants. This means that we need a number of conditions equal to the order of the equation.
As an example, for a first order equation:

$$\begin{cases} f'(x) = g(x) \\ f(x_0) = f_0 \end{cases} \tag{2.44}$$

where x_0 and f_0 are fixed values of the independent variable and of the function f respectively.

2.3.1 Existence and uniqueness

It can be proven that under certain hypotheses of continuity and derivability, the solution to a Cauchy problem exists, and it is unique.

2.4 Ordinary linear differential equations with constant coefficients

(see [Pet66, chap 6, pag 124] [MS95, sec 4B, pag 211])
It is possible to show, using linear algebra, what is the solution of a linear differential equation of order n with constant coefficients.

2.4.1 Homogeneous equations

We first describe the case of homogeneous equations.
The solution will be the linear combination of several exponential functions, and the coefficients of this linear combination will be unknown constants.

Characteristic equation

To explicitly write the solution, we need to "build" an algebraic equation associated to the linear differential equation. We will use as unknown variable of the equation a different letter (e.g. λ), and we will write a term with power k for each term with derivative of order k of the differential equation. As an example, to the equation:

$$af'' + bf' + cf = 0 \tag{2.45}$$

will be associated the equation

$$a\lambda^2 + b\lambda + c = 0 \tag{2.46}$$

which will have the two solutions

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2.47}$$

and all the solutions of the differential equations will be represented as:

$$f(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}. \tag{2.48}$$

Note: The set of all the (infinite) solutions of the linear differential equation with constant coefficients is a vector space, with dimension equal to the order of the equation. The exponentials in the solution (2.48) are a basis of this vector space. It is in principle possible to write the solution using other functions as a basis for the linear combination.
In the case two (or more) solutions of the characteristic equation coincide, in order to write the linear combination (2.48), we will obtain linearly independent exponentials multiplying them for the independent variable:

$$f(x) = C_1 e^{\lambda x} + C_2 x e^{\lambda x}. \tag{2.49}$$

In the case of complex solutions, where $(b^2 - 4ac) < 0$, it is possible to write the solution in a real form using the Euler formula:

$$\begin{cases} e^{ix} = \cos x + i \sin x \\ e^{-ix} = \cos x - i \sin x \end{cases} \tag{2.50}$$

obtaining:

$$f(x) = C_1\ e^{\alpha x} \cos \beta x + C_2\ e^{\alpha x} \sin \beta x, \tag{2.51}$$

where

$$\begin{cases} \alpha = -\dfrac{a}{2} \\ \beta = \dfrac{\sqrt{-(b^2-4ac)}}{2} \end{cases} \tag{2.52}$$

(see also [MS95, page 213, formula (iii)])

2.4.2 Non-homogeneous equations

The first step to solve a non-homogeneous linear differential equation with constant coefficients:

$$f'' + bf' + cf = g(x) \tag{2.53}$$

is to solve the "associated" homogeneous equation.

$$f'' + bf' + cf = 0. \tag{2.54}$$

The general solution of the associated homogeneous equation is a *set of linearly independent functions*, as shown in (2.48):

$$\{f_1(x), f_2(x)\}. \tag{2.55}$$

Then, we need to find one function that satisfy (i.e. is a solution of) the non homogeneous equation:

$$\tilde{f}(x) \text{ such that } \tilde{f}'' + b\tilde{f}' + c\tilde{f} = g(x) \tag{2.56}$$

.
Finally, the general solution (i.e. the set of all the infinite solutions) of the non-homogeneous equation is the sum of the (2.48) plus the (2.56):

$$f(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \tilde{f}(x). \tag{2.57}$$

2.5 Method of Lagrange

(Also called "method of the variation of constants", see [Zil13, pag 156] and [MS95, pag 232])

This method allows to find a particular solution of a non-homogeneous differential equation, after we have found the general solution of the associated homogeneous solution. Notice that here there is no requirement for the coefficients to be constant.
We start with an example of a differential equation of the second order; we will see that this method can be applied to equations of any order.
Let's consider the equation:

$$\frac{d^2 f(x)}{dx^2} + a_1 \frac{df(x)}{dx} + a_0 f(x) = g(x) \tag{2.58}$$

and let's say that the general solution fo the homogeneous associated equation is $\{f_1, f_2\}$.
Then, the first step of the method is to solve the following set of differential equations:

$$\begin{cases} \dfrac{d\gamma_1}{dx} f_1 + \dfrac{d\gamma_2}{dx} f_2 = 0 \\ \dfrac{d\gamma_1}{dx} \dfrac{df_1}{dx} + \dfrac{d\gamma_2}{dx} \dfrac{df_2}{dx} = g \end{cases} \tag{2.59}$$

This is an equation where the unknown functions are $\{\gamma_1(x), \gamma_2(x)\}$, and the two functions $\{f_1(x), f_2(x)\}$ are considered as known.
This system of equations can be solved, to first find the two derivatives $\{\frac{d\gamma_1}{dx}, \frac{d\gamma_2}{dx}\}$.
Once we have the derivatives, we can (hopefully) compute the integrals and find $\{\gamma_1, \gamma_2\}$:

$$\begin{cases} \gamma_1 = \int \frac{d\gamma_1}{dx} dx \\ \gamma_2 = \int \frac{d\gamma_2}{dx} dx \end{cases} \tag{2.60}$$

Once we have found $\{\gamma_1, \gamma_2\}$, we can write the *particular solution* of the non-homogeneous equation (2.58):

$$\tilde{f}(x) = \gamma_1(x) f_1(x) + \gamma_2(x) f_2(x). \tag{2.61}$$

Notice that since in the end we need a linear combination of the functions $\{\gamma_1, \gamma_2\}$, when we compute the integrals (2.60), we can neglect the constants of the indefinite integrals.
Finally, the *general solution* to the non-homogeneous equation will be:

$$\boxed{f(x) = C_1 f_1(x) + C_2 f_2(x) + \tilde{f}(x)} \tag{2.62}$$

2.5.1 Generalization to order n

We have seen the case of an order-2 equation. We can extend to the case of order n:
We will have the *general solution* to the homogeneous equation, which will be the linear combination of n functions:

$$C_1 f_1 + C_2 f_2 + \ldots + C_n f_n \tag{2.63}$$

and then we will need to find n unknown functions $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$
Solving the following system of differential equations:

$$\begin{cases} \frac{d\gamma_1}{dx} f_1 + \frac{d\gamma_2}{dx} f_2 + \ldots + \frac{d\gamma_n}{dx} f_n = 0 \\ \frac{d\gamma_1}{dx}\frac{df_1}{dx} + \frac{d\gamma_2}{dx}\frac{df_2}{dx} + \ldots + \frac{d\gamma_n}{dx}\frac{df_n}{dx} = 0 \\ \ldots \\ \frac{d\gamma_1}{dx}\frac{d^{n-1}f_1}{dx^{n-1}} + \frac{d\gamma_2}{dx}\frac{d^{n-1}f_2}{dx^{n-1}} + \ldots + \frac{d\gamma_n}{dx}\frac{d^{n-1}f_n}{dx^{n-1}} = g. \end{cases} \tag{2.64}$$

The matrix of the coefficients of this system has a name: the **wronskian**:

$$\begin{pmatrix} f_1 & f_2 & \ldots & f_n \\ \frac{df_1}{dx} & \frac{df_2}{dx} & & \frac{df_n}{dx} \\ & \ldots & \ldots & \\ \frac{d^{n-1}f_1}{dx^{n-1}} & \frac{d^{n-1}f_2}{dx^{n-1}} & \ldots & \frac{d^{n-1}f_n}{dx^{n-1}}. \end{pmatrix} \tag{2.65}$$

2.5.2 Exercise

[MS95, pag 233, ex. 4.47]
Using the Lagrange method, let's solve (i.e. find the general solution) the following non-homogeneous equation, *considering that we* ***already have*** *the (general) solution of the homogeneous associated equation*:

$$\frac{d^2 f(x)}{dx^2} - f(x) = 3x^2 - 1 \tag{2.66}$$

where we have the following general solution for the associate homogeneous equation $\frac{d^2 f(x)}{dx^2} - f(x) = 0$:

$$f_h(x) = C_1 e^x + C_2 e^{-x}. \tag{2.67}$$

At first, the Lagrange method consists in solving the following system of equations (2.59), that in this case, considering (2.67), is:

$$\begin{cases} \frac{d\gamma_1}{dx}\, e^x \;+\; \frac{d\gamma_2}{dx}\, e^{-x} \;=\; 0 \\ \frac{d\gamma_1}{dx}\, e^x \;-\; \frac{d\gamma_2}{dx}\, e^{-x} \;=\; 3x^2 - 1 \end{cases} \tag{2.68}$$

So now we solve algebraically this system, for the unknown functions $\frac{d\gamma_{1,2}}{dx}$. It is more effective to consider the derivatives $\frac{d\gamma_2}{dx}$ and $\frac{d\gamma_2}{dx}$ as the unknowns of this system. So, if we start with the first equations we have:, we have:

$$\frac{d\gamma_1}{dx}\, e^x \;+\; \frac{d\gamma_2}{dx}\, e^{-x} \;=\; 0 \tag{2.69a}$$

$$\frac{d\gamma_1}{dx}\, e^x \;=-\; \frac{d\gamma_2}{dx}\, e^{-x} \tag{2.69b}$$

$$\frac{d\gamma_1}{dx} \;=-\; \frac{d\gamma_2}{dx}\, e^{-x}\frac{1}{e^x} \tag{2.69c}$$

$$\frac{d\gamma_1}{dx} \;=-\; \frac{d\gamma_2}{dx}\, e^{-2x} \tag{2.69d}$$

$$\tag{2.69e}$$

and then, inserting this in the second equation:

$$\frac{d\gamma_1}{dx}\, e^x \;-\; \frac{d\gamma_2}{dx}\, e^{-x} \;=\; 3x^2 - 1 \tag{2.70a}$$

$$-\frac{d\gamma_2}{dx}\, e^{-2x} \;-\; \frac{d\gamma_2}{dx}\, e^{-x} \;=\; 3x^2 - 1 \tag{2.70b}$$

$$-\frac{d\gamma_2}{dx}\, e^{-x}\left(e^{-x} + 1\right) \;=\; 3x^2 - 1 \tag{2.70c}$$

$$-\frac{d\gamma_2}{dx}\, \frac{1}{e^x}\left(e^{-x} + 1\right) \;=\; 3x^2 - 1 \tag{2.70d}$$

$$-\frac{d\gamma_2}{dx}\, \left(1 + e^{-x}\right) \;=\; 3x^2 - 1 \tag{2.70e}$$

$$\frac{d\gamma_2}{dx} \;=\; -\frac{3x^2 - 1}{(e^{-x} + 1)} \tag{2.70f}$$

[...] (here I can't reproduce the book's result ☹)
Finally

$$\begin{cases} \frac{d\gamma_1}{dx} = e^{-x}\frac{3x^2 - 1}{2} \\ \frac{d\gamma_2}{dx} = -e^{x}\frac{3x^2 - 1}{2}. \end{cases} \tag{2.71}$$

Then, as usual in this method, the relatively more difficult part is to integrate these functions, to find γ_1 and γ_2.

We can use *two times* the "integration by parts" $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx + c$, and, using $\frac{de^{-x}}{dx} = -e^{-x}$ write:

$$\begin{aligned} \gamma_1 &= \int \frac{3x^2-1}{2} e^{-x} dx \\ &= \frac{1}{2} \int \left(3x^2-1\right) e^{-x} dx = \frac{1}{2} \left[-\left(3x^2-1\right) e^{-x} - \int 6x\ e^{-x} dx \right] \\ &= -\frac{1}{2} \left(3x^2-1\right) e^{-x} - 3 \left[-x\ e^{-x} - \int e^{-x} dx \right] \\ &= e^{-x} \left[-\frac{3}{2} x^2 + \frac{1}{2} + x + 1 \right] = -\frac{1}{2} e^{-x} \left[3x^2 - x - 3 \right] \end{aligned} \tag{2.72}$$

(Again, here I can't get the book's result: $\gamma_1 = -\frac{3}{2} e^{-x} \left[x^2 + 2x + \frac{5}{3} \right]$)

Similarly, applying the same approach we can compute $\gamma_2(x)$.
Finally, the general solution of the equation (2.66) will be:

$$f(x) = C_1 e^x + C_2 e^{-x} - \left[\gamma_1(x) e^x + \gamma_2(x) e^{-x} \right] \tag{2.73}$$

2.6 Fourier theory

This section will introduce the theory of Fourier series and Fourier transforms. Those subjects are very important in many fields of Science, and involve rather deep mathematic knowledge. We will not cover all the details, that go well beyond the scope of the Calculus course.
We will use some of the results from this section, to solve the *heat equation*, in the following section. Solve the heat equation is incidentally the historical reason that lead Joseph Fourier to develop the theory that we will present in this section.

References: [ZC08, cap. 11, pag. 397], [KF61, sec. §54, pag. 96], [KF80, sec. VII §3, pag 383].

2.6.1 Vector space of functions

When we have studied the linear differential equations, we have already seen the concept that a set of functions can be organized as a vector space. At the root of this idea is the fact that a *linear combination of solutions of a given linear differential equation is still a solution of the equation*. Here we want to expand this idea.
We start considering *all the functions defined on the finite interval* $[a, b]$, and we call this set V. Since we can define the linear combination of functions of V, which still belong to V, we can imagine that V is a vector space. We are not going to show rigorously this last statement, just consider that we would need to prove the closure property: any linear combination of elements of V is still an element of V.

Scalar product

Since the vector space V we want to build is euclidean, we want to define a *scalar product between functions*. So, we can define the scalar product between elements of V as:

$$(f, g) = \int_a^b f(x)g(x)dx. \tag{2.74}$$

Once we have a scalar product, we can define *orthogonal functions* as those functions $f(x), g(x) \in V$ such that

$$f(x) \perp g(x) \Leftrightarrow (f, g) = \int_a^b f(x)g(x)dx = 0. \tag{2.75}$$

Norm of a function

We define the norm of a function as the square root of the scalar product of the function times itself:

$$\|f(x)\| \equiv \sqrt{\int_a^b f(x)^2 dx} \tag{2.76}$$

We call *normal* a function that has norm $\|f(x)\| = 1$.
Notice that if the norm of a function $\|f(x)\|$ is different from 1, it is easy to compute a function proportional to that, but with norm one: we compute the norm of the function, and then divide the function by its norm:

$$f_{\text{norm}}(x) = \frac{1}{\|f(x)\|} f(x) \tag{2.77}$$

We call this technique *normalization*.

Orthonormal basis

Then, we can build a set of functions $\{\phi_0, \phi_1, \ldots, \phi_n\}$ that are all orthogonal to each other, and that have all norm=1:

$$(\phi_i, \phi_j) = \delta_{i,j} \tag{2.78}$$

where we have used the symbol $\delta_{i,j}$ which is called Kroneker delta

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j\ . \end{cases} \tag{2.79}$$

Then, our goal is even more ambitious: we want to build a *basis* for our vector space V. So, we need a set of functions of V, with the property that all the function of the set should be linearly independent. Again, we are not going to discuss this part rigorously, we only say that:

- if we want for V the property of *closure*, i.e. that any linear combination of functions of V is still a function of V, then the number of elements of the basis will be *infinite*. This means that the dimension of V is infinite.
- to realize the closure, we need to include special functions, and to include these special functions (the distributions) we need to use a different definition of integral (the Lebesgue integral).

The Fourier theory aims at the realization of a basis for the functions that is *ortonormal*, i.e. we want that any pair of functions of the basis is orthogonal to each other, and that all the functions of the basis are normalized.
We are not going to *derive* the basis of Fourier, we are only going to present it, and to prove that they indeed are orthonormal.

The Fourier set

$$\left\{\frac{1}{2},\ \cos(nx),\ \sin(nx)\right\}_{n=1}^{\infty} \tag{2.80}$$

is such that any function of V can be written as a linear combination of the functions of the Fourier set (2.80) .
It is possible to verify that each function of the set is orthogonal to each other.
In this form, the functions are not normalized; a normalized version is:
[...]

2.6.2 The Fourier series

Infinite linear combination

If we write formally the linear combination of the "basis" functions in (2.80) :

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \tag{2.81}$$

we can see this as *a method to write any function (defined on the interval* $[a, b]$*)* as a sum of the "Fourier functions". This can be useful in many circumstances. As an example, the trigonometric functions can be easily computed as solutions of a differential equation. Then we can write the initial/boundary conditions as linear combinations of trigonometric functions, and find the general solution to the differential equation.

Intuitive interpretation

[...]

2.7 Partial differential equations

Sometimes the name "Partial Differential Equation" is shortened as PDE. Similarly, the name "Ordinary Differential Equation" is shortened as ODE.

2.7.1 Laplacian operator

Let's consider a function of time and space $f(t, x_1, x_2, x_3)$. We define the laplacian operator ∇^2 as follows:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \tag{2.82}$$

so that:

$$\nabla^2 f(x_1, \ldots, x_n) = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}. \tag{2.83}$$

The previous is the most general definition, in n variables. Of course, this operator can be defined in particular in the case where the spatial variables are three, or two, or just one.

2.7.2 Heat propagation equation

See [ZC08, chap. 12]
(An interesting video lecture on this subject can be found here:
https://www.youtube.com/watch?v=ToIXSwZ1pJU, while this is the link to the full playlist, about differential equations).
Using the operator (2.82), we can introduce an important differential equation:

$$\boxed{\frac{\partial f(\vec{x}, t)}{\partial t} + d\nabla^2 f(\vec{x}, t) = g(\vec{x}, t)} \tag{2.84}$$

This equation is very important, has many "applications" in different fields of science.
The full discussion of this equation, and its solutions, needs advanced mathematical subjects, so, in our course we will inly look at some parts of the full discussion, and in some points we will take some "shortcuts", that will be highlighted.
Let's start to discuss a simple case, where the spatial part is unidimensional: $\vec{x} \to x$. Moreover, to make the equation even more simple, we assume $d = 1$. In this case the equation is:

$$\frac{\partial f(t,x)}{\partial t} - \frac{\partial^2 f(t,x)}{\partial x^2} = 0 \tag{2.85}$$

Here we will discuss the solution of a Cauchy problem, i.e. the solution of the differential equation together with the initial condition and boundary conditions. In this case the solution is a function $f(x,t)$:

$$\begin{cases} \dfrac{\partial f(t,x)}{\partial t} - \dfrac{\partial^2 f(t,x)}{\partial x^2} = 0 \\ f(0,x) = x \\ f(t,0) = f(t,1) = 0 \end{cases} \tag{2.86}$$

(here I am following the computation from the italian wikiedia page, here is the translation in english obtained with 'google translate')
First, we apply the *method of separation of variables*. This method consists in **assuming the hypothesis** that the solution of this PDE can be written as a product of two parts, one depending only on time, and one depending only on position:

$$f(t,x) = T(t) \cdot X(x). \tag{2.87}$$

If we plug this hypothesis in the equation we have:

$$\frac{\partial T(t)\ X(x)}{\partial t} - \frac{\partial^2 T(t)\ X(x)}{\partial x^2} = 0 \tag{2.88a}$$

$$X(x)\frac{dT(t)}{dt} - T(t)\frac{d^2 X(x)}{dx^2} = 0 \tag{2.88b}$$

here we have brought outside the derivative the factors that do not depend on the variable of the derivative (they are *constant*, with respect to that derivative. Moreover, the derivative is now applied to a function that only depends on one variable, so they are no more partial derivatives, but *total* derivatives.
Now, we can group the spatial terms and the temporal terms separately:

$$X(x)\frac{dT(t)}{dt} = T(t)\frac{d^2 X(x)}{dx^2} \tag{2.89a}$$

$$\frac{1}{T(t)}\frac{dT(t)}{dt} = \frac{1}{X(x)}\frac{d^2 X(x)}{dx^2}. \tag{2.89b}$$

Let's observe this last equation: we have a function on the left, which must be equal to a function on the right. However, the function on the left only depends on t, and only on t, and the function on the right depends on x, and only on x. Still, they must be equal, for any value of x and t. The only possible situation that satisfies this request is that both terms are equal to a constant, the same constant that we will call λ, that is constant with respect to t and x:

$$\begin{cases} \dfrac{1}{T(t)}\dfrac{dT(t)}{dt} = \lambda \\ \dfrac{1}{X(x)}\dfrac{d^2 X(x)}{dx^2} = \lambda \end{cases} \tag{2.90}$$

So, assuming the initial hypothesis $f(t,x) = T(t) \cdot X(x)$, we have turned the PDE into a system of two ODE, one in t and one in x.

Solution of temporal part

Let's work first on the first equation:

$$\begin{aligned} &\frac{1}{T(t)}\frac{dT(t)}{dt} = \lambda \\ &\frac{dT(t)}{dt} = \lambda T(t) \end{aligned} \tag{2.91}$$

We can solve "immediately" this differential equation just using the formula for the derivative of the "composite exponential": $\frac{de^{\eta(t)}}{dt} = e^{\eta(t)}\frac{d\eta(t)}{dt}$, and in particular, since in our case is $\eta(t) = \lambda t$, and $\frac{d\lambda t}{dt} = \lambda$, we have

$$T(t) = e^{\lambda t}C \tag{2.92}$$

where C is the usual constant coming from the indefinite integration. To assign a value to this integration constant C we must consider the initial condition. However, the initial condition is expressed in terms of the full function $f(t,x)$, where here we need the initial value for the "factor" $T(t)$. So here we will use a temporary expression $T(0)$, that we need to express explicitly later:

$$T(t) = T(0)e^{\lambda t}. \tag{2.93}$$

Solution of the spatial part

Let's now look at the spatial ODE:

$$\begin{aligned} &\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \lambda \\ &\frac{d^2X(x)}{dx^2} = \lambda X(x) \\ &\frac{d^2X(x)}{dx^2} - \lambda X(x) = 0. \end{aligned} \tag{2.94}$$

We need to solve this equation together with the boundary conditions:

$$\begin{cases} \frac{d^2X(x)}{dx^2} - \lambda X(x) = 0 \\ X(0) = X(1) = 0 \end{cases} \tag{2.95}$$

To solve this equation we will use the theory of Fourier. Indeed, historically Fourier developed his theory to solve the heat equation.
We start noticing that a $\sin(x)$ function would satisfy this equation, since $\frac{d\sin(x)}{dx} = \cos(x)$, and $\frac{d\cos(x)}{dx} = -\sin(x)$ we have:

$$\frac{d^2\sin(x)}{dx^2} = -\sin(x). \tag{2.96}$$

We can also notice that any other $\sin(nx)$ function with a different period, would be a solution for this equation

2.8 Appendix on Fourier theory

In this section I copy old notes that I have on Fourier theory. I have not checked deeply this material, but it should be ok. I will incorporate in the section on Fourier theory later.

Chapter 3

Fourier Theory

3.1 Periodic functions

3.1.1 functions with period 2π

Let's consider the space $\mathcal{L}_2(-\pi, \pi)$ of the functions "square-Lebesgue-integrable" over $(-\pi, \pi)$. This space is:

- euclidean
- complete
- of infinite dimension

and therefore it is an *Hilbert space*.
We note that using the *extention by continuity* we can extend what we find for the function defined in $(-\pi, \pi)$, to the functions defined on $\mathbb{R}$, bounded and periodic, with period 2π.

trigonometric form

It is possible to show that a basis for this space is:

$$\{1, \cos(kx), \sin(kx)\}_{k=1}^{\infty} \tag{3.1}$$

If we expand one of such functions on this basis we can write:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \tag{3.2}$$

where we have defined

$$a_k \equiv \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt) \mathrm{d}t \tag{3.3a}$$

$$b_k \equiv \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt) \mathrm{d}t \; . \tag{3.3b}$$

In [KF80] can be found theorems which specify the hypotheses for $f(x)$ that guarantee the convergence of the sum, and the fact that it converges at an unique function.

complex form

(see [KF80, cap. VII, §3] (Cap VII Spaces of integrable functions, §3. Sistems of orthogonal functions in $\mathcal{L}_2$. Series with respect to orthogonal systems)) Instead of using the basis (3.1), we can use the basis:

$$\{e^{ikx}\}_{k=-\infty}^{+\infty} \tag{3.4}$$

The basis (3.4) is obtained from (3.1) just applying Euler relations:

$$\cos(kx) = \frac{e^{ikx} + e^{-ikx}}{2} \tag{3.5a}$$

$$\sin(kx) = \frac{e^{ikx} - e^{-ikx}}{2i} \tag{3.5b}$$

Then, the Fourier series using this basis becomes:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} c_k \; e^{ikx}} \tag{3.6}$$

where the coefficients c_k can be expressed as functions of the coefficients a_k and b_k of the trigonometric form (again using Euler's relations):

$$c_0 = \frac{a_0}{2} \tag{3.7a}$$

$$c_k = \frac{a_k - ib_k}{2} \tag{3.7b}$$

$$c_{-k} = \frac{a_k + ib_k}{2} \tag{3.7c}$$

and then explicitly:

$$\boxed{c_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) \; e^{-ikx} \mathrm{d}x} \tag{3.8}$$

3.1.2 Functions with period $2l$

Let's now consider the space $\mathcal{L}_2(-l, l)$ Hilbert space of the functions "square-Lebesgue-integrable" over $(-l, l)$. Be the function $f(x)$ an element of such space defined on a finite interval $(-l, l)$.
Again, using the *extention by continuity* we can also consider the functions defined on $\mathbb{R}$, bounded and periodic, with period $2l$.

trigonometric form

It is possible to show that a basis for the space $\mathcal{L}_2(-l, l)$ is:

$$\left\{1, \cos\left(k\frac{\pi}{l}x\right), \sin\left(k\frac{\pi}{l}x\right)\right\}_{k=1}^{\infty} \tag{3.9}$$

Let's suppose this function satisfy the hypotheses under which the Fourier series converges. Then, in analogy to (3.2) we can write:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(k\frac{\pi}{l}x\right) + b_k \sin\left(k\frac{\pi}{l}x\right) \tag{3.10}$$

where we have defined

$$a_k \equiv \frac{1}{l} \int_{-l}^{l} f(t) \cos\left(k\frac{\pi}{l}t\right) \mathrm{d}t \tag{3.11a}$$

$$b_k \equiv \frac{1}{l} \int_{-l}^{l} f(t) \sin\left(k\frac{\pi}{l}t\right) \mathrm{d}t \; . \tag{3.11b}$$

Then we can rewrite (3.10) with the explicit (3.11) as:

$$\begin{aligned} f(x) = \frac{1}{2l} \int_{-} f(t)\mathrm{d}t \quad + \sum_{k=1}^{\infty} \left[\frac{1}{l} \int_{-} f(t) \cos\left(k\frac{\pi}{l}t\right) \mathrm{d}t \;\; \cos\left(k\frac{\pi}{l}x\right) + \right. \\ \left. + \frac{1}{l} \int_{-} f(t) \sin\left(k\frac{\pi}{l}t\right) \mathrm{d}t \;\; \sin\left(k\frac{\pi}{l}x\right)\right] \end{aligned} \tag{3.12}$$

$$f(x) = \frac{1}{2l}\int_{-} f(t)\mathrm{d}t \quad + \sum_{k=1}^{\infty}\left[\frac{1}{l}\int_{-} f(t)\cos\left(k\frac{\pi}{l}t\right)\cos\left(k\frac{\pi}{l}x\right)\mathrm{d}t + \right. \\ \left. + \frac{1}{l}\int_{-} f(t)\sin\left(k\frac{\pi}{l}t\right)\sin\left(k\frac{\pi}{l}x\right)\mathrm{d}t\right] \tag{3.13}$$

where we have included the $\cos(kx)$ and $\sin(kx)$ since they do not depend on the integrating variable t. Merging the two integrals we have:

$$f(x) = \frac{1}{2l}\int_{-} f(t)\mathrm{d}t + \sum_{k=1}^{\infty}\frac{1}{l}\int_{-} f(t)\left[\cos\left(k\frac{\pi}{l}t\right)\cos\left(k\frac{\pi}{l}x\right) + \sin\left(k\frac{\pi}{l}t\right)\sin\left(k\frac{\pi}{l}x\right)\right]\mathrm{d}t \tag{3.14}$$

Then we use the product-to-sum trigonometric identities

$$\sin A \sin B = (1/2)[\cos(A-B) - \cos(A+B)] \tag{3.15a}$$

$$\cos A \cos B = (1/2)[\cos(A-B) + \cos(A+B)] \tag{3.15b}$$

$$\Downarrow$$

$$\sin A \sin B + \cos A \cos B = \cos(A-B) \tag{3.15c}$$

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\mathrm{d}t + \sum_{k=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(t)\cos\left[k\frac{\pi}{l}(t-x)\right]\ \mathrm{d}t. \tag{3.16}$$

complex form

In analogy with what we have done in subsection 3.1.1, we can use the Euler's relations and re-write the basis (3.9)

$$\{e^{ik\frac{\pi}{l}x}\}_{k=-\infty}^{\infty} \tag{3.17}$$

and then (3.10) becomes

$$f(x) = \sum_{k=-\infty}^{\infty} c_k\ e^{k\frac{\pi}{l}x} \tag{3.18}$$

where the coefficients are defined as

$$c_k \equiv \frac{1}{l}\int_{-l}^{l} f(t)\ e^{-ik\frac{\pi}{l}t}\ \mathrm{d}t \tag{3.19}$$

3.2 Functions defined on $\mathbb{R}$

We want to extend the results to the space of $\mathcal{L}_2(-\infty,\infty)$, the Hilbert space of the functions "square-Lebesgue-integrable" over $\mathbb{R}$. As a first approach we consider a non-rigorous limit $l \to \infty$ of (3.16).
First, we introduce the quantities:

$$\lambda_k \equiv k\frac{\pi}{l} \tag{3.20a}$$

$$\Delta\lambda \equiv \frac{\pi}{l} \tag{3.20b}$$

3.2.1 trigonometric form

We can re-write (3.16) using (3.20) as:

$$\begin{aligned} f(x) &= \frac{1}{2l}\int_{-l}^{l} f(t)\mathrm{d}t + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\pi}{l}\int_{-l}^{l} f(t)\cos\left[k\frac{\pi}{l}(t-x)\right]\ \mathrm{d}t \\ &= \frac{1}{2l}\int_{-l}^{l} f(t)\mathrm{d}t + \frac{1}{\pi}\sum_{k=1}^{\infty}\Delta\lambda\int_{-l}^{l} f(t)\cos\left[\lambda_k(t-x)\right]\ \mathrm{d}t. \end{aligned} \tag{3.21}$$

Then we notice that in the hypothesis $\int_{-\infty}^{\infty}|f(t)|\,\mathrm{d}t < \infty$ it is:

$$\lim_{l\to\infty}\frac{1}{2l}\int_{-l}^{l} f(t)\mathrm{d}t = 0. \tag{3.22}$$

Looking at the second term, we see the infinite sum, where the "running index" is k. As this running integer index takes its values from 1 to ∞, we can consider $\lambda_k = k\frac{\pi}{l}$ as an "independent variable", which takes its discrete values, each at a constant distance $\Delta\lambda = \frac{\pi}{l}$ from one another. As the l parameter grows , the independent discrete variable λ_k become more and more similar to a continuous variable:

$$\lim_{l\to\infty}\lambda_k = \lambda \tag{3.23}$$

and the constant size of the "step" $\Delta\lambda$ becomes smaller and smaller, as an infinitesimal differential:

$$\lim_{l\to\infty}\Delta\lambda = \mathrm{d}\lambda. \tag{3.24}$$

Being the sum of a continuous variable with differential steps, and taking into account (3.22), we then re-write (3.21) in the limit $l\to\infty$ as:

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos\left[\lambda(t-x)\right]\ \mathrm{d}t\right]\mathrm{d}\lambda\ . \tag{3.25}$$

Now, we consider that as a function of λ, the expression in the bigger square bracket (i.e. the integrand in $\mathrm{d}\lambda$) is an *even* function of λ (is a cos). And since the integral from 0 to $+\infty$ of an even function is a half of the integral of the same function from $-\infty$ to ∞, we can rewrite:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos\left[\lambda(t-x)\right]\ \mathrm{d}t\right]\mathrm{d}\lambda \tag{3.26a}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cos\left[\lambda(t-x)\right]\ \mathrm{d}t\right]\mathrm{d}\lambda\ . \tag{3.26b}$$

3.2.2 complex form

Since sin is an *odd* function, we can also write:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\sin\left[\lambda(t-x)\right]\ \mathrm{d}t\right]\mathrm{d}\lambda = 0 \tag{3.27}$$

and, multiplying for $-i$, still:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -if(t)\sin\left[\lambda(t-x)\right]\ \mathrm{d}t\right]\mathrm{d}\lambda = 0 \tag{3.28}$$

Then we can add it to (3.29a) without altering it:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cos\left[\lambda(t-x)\right] \; \mathrm{d}t \right] \mathrm{d}\lambda \tag{3.29a}$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -i f(t) \sin\left[\lambda(t-x)\right] \; \mathrm{d}t \right] \mathrm{d}\lambda \tag{3.29b}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left\{ \cos\left[\lambda(t-x)\right] - i \sin\left[\lambda(t-x)\right] \right\} \; \mathrm{d}t \right] \mathrm{d}\lambda \, . \tag{3.29c}$$

Now, using the Euler's formula

$$\cos\theta - i\sin\theta = e^{-i\theta} \tag{3.30}$$

we have:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \; e^{-i\lambda(t-x)} \; \mathrm{d}t \right] \mathrm{d}\lambda \tag{3.31a}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \; e^{-i\lambda t} \; e^{i\lambda x} \; \mathrm{d}t \right] \mathrm{d}\lambda \tag{3.31b}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \; e^{-i\lambda t} \; \mathrm{d}t \right] \; e^{i\lambda x} \; \mathrm{d}\lambda \tag{3.31c}$$

where we have taken the part of the exponential non depending on t out of the inner integral. Finally, we can see the expression in square brackets as a coefficient in the outer integral, and define it as:

$$\boxed{F[f](\lambda) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \; e^{-i\lambda t} \; \mathrm{d}t} \tag{3.32}$$

so to write:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f](\lambda) \; e^{i\lambda x} \; \mathrm{d}\lambda} \tag{3.33}$$

3.2.3 Comments

In passing from the case of periodic functons, to the case of generic, non-periodic functions, we have used a non-rigorous approach, imagining to start with a period $2l$, and then consider the limit $l \to \infty$. For a periodic function, the Fourier expansion is a series, over infinite terms. For each term, the frequency of the oscillating function (element of the Fourier basis, also known as a *mode*) changes of a fixed amount, $\frac{\pi}{l}$. This frequency increment is inversely proportional to the (half) period l. It means that going from periodic to non-periodic functions, in the frequencies domain we go from discrete to continuous changes. We remark that in both cases the frequencies involved are infinite, i.e. the sum over the frequencies has infinite terms. In figure 3.1 a drawing shows the link between the frequencies domain and the "direct" domain.

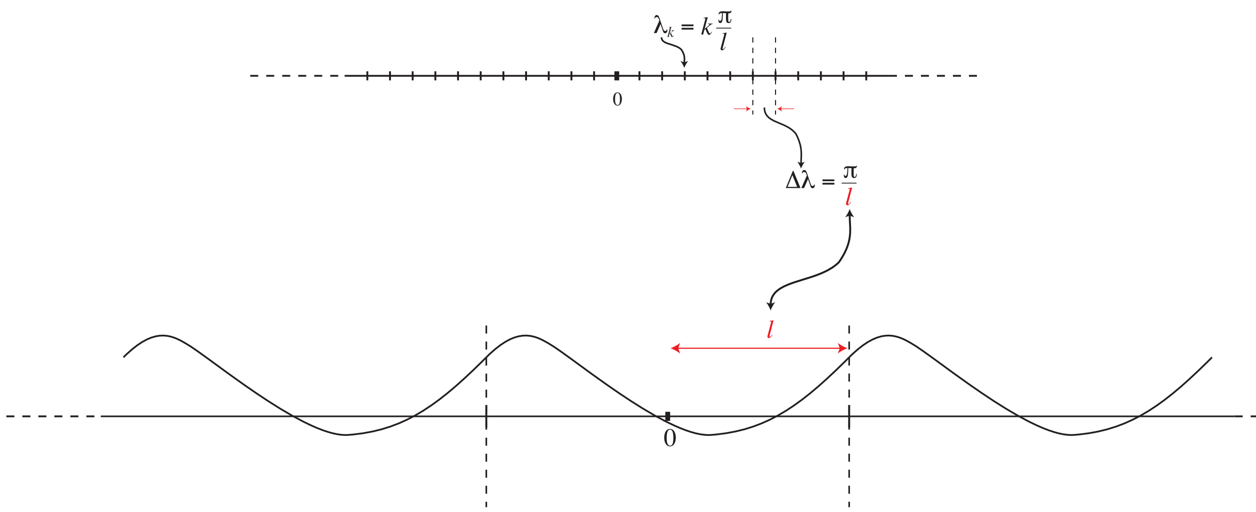

Figure 3.1: l is the (half) period

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