Notes on motion of variable mass systems

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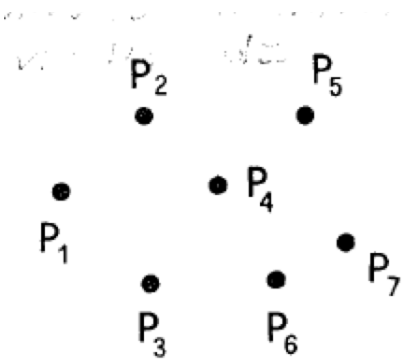
Contents

1 Extended systems

[MS, pag. 203]
A big part of General Physics is discussed using the *dimension-less mathematical point* as model of the system under study.
But this is a limited (although useful) model.
When the system under study can not be modelled as a mathematical point, we upgrade our description, using a set of many mathematical points. This can be as a set of discrete points:



or continuous:

It is easier to discuss the discrete case, where we have a certain number of point-like particles $\{p_i\}$.
For each of these particles, identified by the index "i" we define:

the *momentum*:

$$\vec{q}_i = m_i \vec{v}_i \tag{1}$$

and the *angular momentum*:

$$\vec{p}_i = \Omega\vec{P}_i \times m_i \vec{v}_i \tag{2}$$

where m_i is the mass, $\vec{v}_i$ is the speed, and $\vec{r}_{\Omega i}$ is the position of the particle with respect to a point Ω called "pole", common for all the particles, and chosen arbitrarily.

1.1 Dynamics equations

For each point we can write some equations: the second principle of dynamics (f=ma), where we write the second term as the derivative of the momentum

$$\vec{f}_i = \frac{d\vec{q}_i}{dt} \tag{3}$$

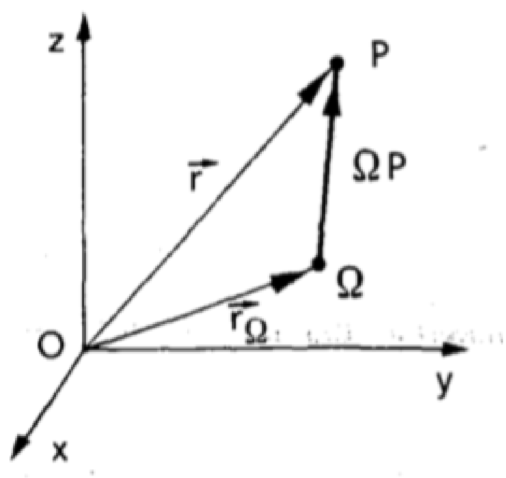
where $\vec{f}_i$ is the (vectorial) sum of all the forces acting on the point i. then we divide the forces into two groups: those due to the interaction with other points of the extended body (internal forces):

$$\vec{f}_i^{(\mathrm{ext})} + \vec{f}_i^{(\mathrm{int})} = \frac{d\vec{q}_i}{dt}. \tag{4}$$

Another equation that we can write is the definition of *torque*, sometimes also called *force momentum*, or *momentum of force*:

$$\vec{m}_i \equiv \Omega P_i \times \vec{f} \tag{5}$$

where $\vec{f}$ is the force acting on the point P_i,
ΩP_i is the *position of the point* P_i *with respect to a reference point* Ω, called "pole",



and $\times$ is the *cross product* or *vector product*.
If we use the (4) into the (6) we have:

$$\vec{m}_i = \Omega P_i \times \frac{d\vec{q}}{dt} \tag{6}$$

and then, using the definition of *angular momentum* (2) $\vec{p} = \Omega P \times \vec{q}$ we can write:

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(\Omega P_i \times \vec{q}) \tag{7a}$$
$$= \frac{d\Omega P_i}{dt} \times \vec{q} + \Omega P_i \times \frac{d\vec{q}}{dt} \tag{7b}$$

where we have used the chain rule for the derivative of a (vector) product.
So, substituing in (6) we have

$$\vec{m}_i = \frac{d\vec{p}}{dt} - \frac{d\Omega P_i}{dt} \times \vec{q} \tag{8}$$

And finally, re-writing $\Omega P_i = \vec{r} - \vec{r}_\Omega$ we have

$$\vec{m}_i = \frac{d\vec{p}}{dt} - (\vec{v} - \vec{v}_\Omega) \times \vec{q} \tag{9}$$

then, observing that the velocity of the point P and its momentum are parallel, we have $\vec{v} \times \vec{q} = 0$ we write:

$$\vec{m}_i = \frac{d\vec{p}}{dt} - +\vec{v}_\Omega \times \vec{q} \tag{10}$$

1.2 Summary of dynamics equations

In summary, the equations that describe the extended systems dynamics are:

$$\vec{f}_i^{(\text{ext})} + \vec{f}_i^{(\text{int})} = \frac{d\vec{q}_i}{dt}. \tag{11}$$

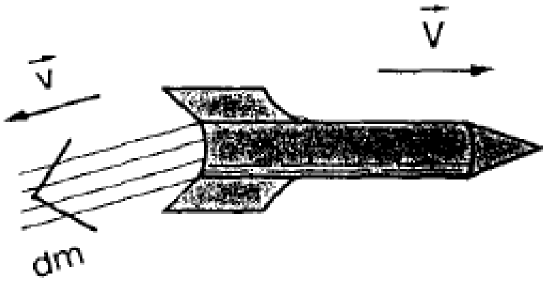
$$\vec{m}_i^{(\text{ext})} + \vec{m}_i^{(\text{int})} = \frac{d\vec{p}_i}{dt} - \vec{v}_\Omega \times \vec{q}_i \tag{12}$$

2 The variable mass problem

[MS, pag. 223]
We want to “solve” a dynamic system where the mass is not constant. (Note: “solve” a dynamic system usually means to start with the equations of forces and accelerations, which are differential equations, and then integrate (solve) them, to find the “time law”, i.e. the position as a function of time.)
So, we start with equation (4):

$$\vec{F}^{\text{ext}} = \frac{d\vec{Q}}{dt} \tag{13}$$

where $Q(t) = M(t)V(t)$ is the momentum of the (total) system.
To prepare to solve the equation (13) we want to write $Q(t)$ at time $t + \Delta t$.



$$\vec{Q}(t + \Delta t) = (M(t) - dm)\vec{V}(t + \Delta t) + dm\vec{v} \tag{14a}$$

where:

- $(M(t) - dm)$ is the mass of the rocket minus the mass of the expelled fuel;
- $\vec{V}(t + \Delta t)$ is the speed of the rocket at time $t + \Delta t$;
- $dm\vec{v}$ is the momentum of the expelled fuel (the direction of this speed is the opposite of the direction of rocket's speed).

Once we have $\vec{Q}(t + \Delta t)$, we can write the variation of $\vec{Q}$, written as $\Delta\vec{Q} = \vec{Q}(t + \Delta t) - \vec{Q}(t)$:

$$\Delta\vec{Q} = \vec{Q}(t + \Delta t) - \vec{Q}(t) \tag{15a}$$
$$= (M(t) - dm)\vec{V}(t + \Delta t) \; + \; dm\;\vec{v} - M(t)\vec{V}(t) \tag{15b}$$
$$= M(t)\left[\vec{V}(t + \Delta t) - \vec{V}(t)\right] \; + \; dm\left[\vec{v} - \vec{V}(t + \Delta t)\right] \tag{15c}$$
$$= M(t)\Delta\vec{V} + \tag{15d}$$

Once we have written $\Delta\vec{Q}$, we can write the derivative $\frac{d\vec{Q}}{dt}$ as $\lim_{\Delta t \to 0} \frac{\Delta\vec{Q}}{\Delta t}$:

$$\frac{\Delta\vec{Q}}{\Delta t} = \left\{M(t)\left[\vec{V}(t + \Delta t) - \vec{V}(t)\right] + dm\left[\vec{v} - \vec{V}(t + \Delta t)\right]\right\} \cdot \frac{1}{\Delta t} \tag{16a}$$
$$= \left\{M(t)\Delta\vec{V}(t) + dm\left[\vec{v} - \vec{V}(t + \Delta t)\right]\right\} \cdot \frac{1}{\Delta t} \tag{16b}$$

References

[MS] Corrado Mencuccini and Vittorio Silvestrini. *Fisica I (Meccanica e termodinamica)*, volume 1. Liguori, Napoli.