Fourier theory

Notes for the Calculus course 2020-2021 at X-Bio - University of Tyumen
(old notes, to be checked)

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Contents

1 Periodic functions

1.1 functions with period 2π

Let's consider the space $\mathcal{L}_2(-\pi, \pi)$ of the functions "square-Lebesgue-integrable" over $(-\pi, \pi)$. This space is:

- euclidean
- complete
- of infinite dimension

and therefore it is an *Hilbert space*.
We note that using the *extention by continuity* we can extend what we find for the function defined in $(-\pi, \pi)$, to the functions defined on $\mathbb{R}$, bounded and periodic, with period 2π.

1.1.1 trigonometric form

It is possible to show that a basis for this space is:

$$\{1, \cos(kx), \sin(kx)\}_{k=1}^{\infty} \tag{1}$$

If we expand one of such functions on this basis we can write:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \tag{2}$$

where we have defined

$$a_k \equiv \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt) \mathrm{d}t \tag{3a}$$

$$b_k \equiv \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt) \mathrm{d}t \ . \tag{3b}$$

In [1] can be found theorems which specify the hypotheses for $f(x)$ that guarantee the convergence of the sum, and the fact that it converges at an unique function.

1.1.2 complex form

(see [1, cap. VII, §3] (Cap VII Spaces of integrable functions, §3. Sistems of orthogonal functions in $\mathcal{L}_2$. Series with respect to orthogonal systems)) Instead of using the basis (1), we can use the basis:

$$\{e^{ikx}\}_{k=-\infty}^{+\infty} \tag{4}$$

The basis (4) is obtained from (1) just applying Euler relations:

$$\cos(kx) = \frac{e^{ikx} + e^{-ikx}}{2} \tag{5a}$$

$$\sin(kx) = \frac{e^{ikx} - e^{-ikx}}{2i} \tag{5b}$$

Then, the Fourier series using this basis becomes:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} c_k \; e^{ikx}} \tag{6}$$

where the coefficients c_k can be expressed as functions of the coefficients a_k and b_k of the trigonometric form (again using Euler's relations):

$$c_0 = \frac{a_0}{2} \tag{7a}$$

$$c_k = \frac{a_k - ib_k}{2} \tag{7b}$$

$$c_{-k} = \frac{a_k + ib_k}{2} \tag{7c}$$

and then explicitly:

$$\boxed{c_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) \; e^{-ikx} \mathrm{d}x} \tag{8}$$

1.2 Functions with period $2l$

Let's now consider the space $\mathcal{L}_2(-l, l)$ Hilbert space of the functions "square-Lebesgue-integrable" over $(-l, l)$. Be the function $f(x)$ an element of such space defined on a finite interval $(-l, l)$.
Again, using the *extention by continuity* we can also consider the functions defined on $\mathbb{R}$, bounded and periodic, with period $2l$.

1.2.1 trigonometric form

It is possible to show that a basis for the space $\mathcal{L}_2(-l, l)$ is:

$$\left\{1, \cos\left(k\frac{\pi}{l}x\right), \sin\left(k\frac{\pi}{l}x\right)\right\}_{k=1}^{\infty} \tag{9}$$

Let's suppose this function satisfy the hypotheses under which the Fourier series converges. Then, in analogy to (2) we can write:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(k\frac{\pi}{l}x\right) + b_k \sin\left(k\frac{\pi}{l}x\right) \tag{10}$$

where we have defined

$$a_k \equiv \frac{1}{l} \int_{-l}^{l} f(t) \cos\left(k\frac{\pi}{l}t\right) \mathrm{d}t \tag{11a}$$

$$b_k \equiv \frac{1}{l} \int_{-l}^{l} f(t) \sin\left(k\frac{\pi}{l}t\right) \mathrm{d}t \; . \tag{11b}$$

Then we can rewrite (10) with the explicit (11) as:

$$\begin{aligned} f(x) = \frac{1}{2l} \int_{-l}^{l} f(t) \mathrm{d}t \quad + \sum_{k=1}^{\infty} \Bigg[\frac{1}{l} \int_{-l}^{l} f(t) \cos\left(k\frac{\pi}{l}t\right) \mathrm{d}t \; \cos\left(k\frac{\pi}{l}x\right) + \\ + \frac{1}{l} \int_{-l}^{l} f(t) \sin\left(k\frac{\pi}{l}t\right) \mathrm{d}t \; \sin\left(k\frac{\pi}{l}x\right) \Bigg] \end{aligned} \tag{12}$$

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\mathrm{d}t \quad + \sum_{k=1}^{\infty}\left[\frac{1}{l}\int_{-l}^{l} f(t)\cos\left(k\frac{\pi}{l}t\right)\cos\left(k\frac{\pi}{l}x\right)\mathrm{d}t + \right. \\ \left. + \frac{1}{l}\int_{-l}^{l} f(t)\sin\left(k\frac{\pi}{l}t\right)\sin\left(k\frac{\pi}{l}x\right)\mathrm{d}t\right] \tag{13}$$

where we have included the $\cos(kx)$ and $\sin(kx)$ since they do not depend on the integrating variable t. Merging the two integrals we have:

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\mathrm{d}t + \sum_{k=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(t)\left[\cos\left(k\frac{\pi}{l}t\right)\cos\left(k\frac{\pi}{l}x\right) + \sin\left(k\frac{\pi}{l}t\right)\sin\left(k\frac{\pi}{l}x\right)\right]\mathrm{d}t \tag{14}$$

Then we use the product-to-sum trigonometric identities

$$\sin A \sin B = (1/2)[\cos(A-B) - \cos(A+B)] \tag{15a}$$

$$\cos A \cos B = (1/2)[\cos(A-B) + \cos(A+B)] \tag{15b}$$

$$\Downarrow$$

$$\sin A \sin B + \cos A \cos B = \cos(A-B) \tag{15c}$$

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\mathrm{d}t + \sum_{k=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(t)\cos\left[k\frac{\pi}{l}(t-x)\right]\mathrm{d}t. \tag{16}$$

1.2.2 complex form

In analogy with what we have done in subsection 1.1.2, we can use the Euler's relations and re-write the basis (9)

$$\left\{e^{ik\frac{\pi}{l}x}\right\}_{k=-\infty}^{\infty} \tag{17}$$

and then (10) becomes

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \; e^{k\frac{\pi}{l}x} \tag{18}$$

where the coefficients are defined as

$$c_k \equiv \frac{1}{l}\int_{-l}^{l} f(t)\; e^{-ik\frac{\pi}{l}t}\;\mathrm{d}t \tag{19}$$

2 Functions defined on $\mathbb{R}$

We want to extend the results to the space of $\mathcal{L}_2(-\infty,\infty)$, the Hilbert space of the functions "square-Lebesgue-integrable" over $\mathbb{R}$. As a first approach we consider a non-rigorous limit $l \to \infty$ of (16).
First, we introduce the quantities:

$$\lambda_k \equiv k\frac{\pi}{l} \tag{20a}$$

$$\Delta\lambda \equiv \frac{\pi}{l} \tag{20b}$$

2.1 trigonometric form

We can re-write (16) using (20) as:

$$\begin{aligned} f(x) &= \frac{1}{2l}\int_{-l}^{l} f(t)\mathrm{d}t + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\pi}{l}\int_{-l}^{l} f(t)\cos\left[k\frac{\pi}{l}(t-x)\right]\ \mathrm{d}t \\ &= \frac{1}{2l}\int_{-l}^{l} f(t)\mathrm{d}t + \frac{1}{\pi}\sum_{k=1}^{\infty}\Delta\lambda\int_{-l}^{l} f(t)\cos\left[\lambda_k(t-x)\right]\ \mathrm{d}t. \end{aligned} \tag{21}$$

Then we notice that in the hypothesis $\int_{-\infty}^{\infty}|f(t)|\,\mathrm{d}t < \infty$ it is:

$$\lim_{l\to\infty}\frac{1}{2l}\int_{-l}^{l} f(t)\mathrm{d}t = 0. \tag{22}$$

Looking at the second term, we see the infinite sum, where the "running index" is k. As this running integer index takes its values from 1 to ∞, we can consider $\lambda_k = k\frac{\pi}{l}$ as an "independent variable", which takes its discrete values, each at a constant distance $\Delta\lambda = \frac{\pi}{l}$ from one another. As the l parameter grows , the independent discrete variable λ_k become more and more similar to a continuous variable:

$$\lim_{l\to\infty}\lambda_k = \lambda \tag{23}$$

and the constant size of the "step" $\Delta\lambda$ becomes smaller and smaller, as an infinitesimal differential:

$$\lim_{l\to\infty}\Delta\lambda = \mathrm{d}\lambda. \tag{24}$$

Being the sum of a continuous variable with differential steps, and taking into account (22), we then re-write (21) in the limit $l \to \infty$ as:

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos\left[\lambda(t-x)\right]\ \mathrm{d}t\right]\mathrm{d}\lambda\ . \tag{25}$$

Now, we consider that as a function of λ, the expression in the bigger square bracket (i.e. the integrand in $\mathrm{d}\lambda$) is an *even* function of λ (is a cos). And since the integral from 0 to $+\infty$ of an even function is a half of the integral of the same function from $-\infty$ to ∞, we can rewrite:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos\left[\lambda(t-x)\right]\ \mathrm{d}t\right]\mathrm{d}\lambda \tag{26a}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cos\left[\lambda(t-x)\right]\ \mathrm{d}t\right]\mathrm{d}\lambda\ . \tag{26b}$$

2.2 complex form

Since sin is an *odd* function, we can also write:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\sin\left[\lambda(t-x)\right]\ \mathrm{d}t\right]\mathrm{d}\lambda = 0 \tag{27}$$

and, multiplying for $-i$, still:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}-if(t)\sin\left[\lambda(t-x)\right]\ \mathrm{d}t\right]\mathrm{d}\lambda=0 \tag{28}$$

Then we can add it to (29a) without altering it:

$$\begin{aligned}
f(x)= &\quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\cos\left[\lambda(t-x)\right]\ \mathrm{d}t\right]\mathrm{d}\lambda && \text{(29a)}\\
&+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}-if(t)\sin\left[\lambda(t-x)\right]\ \mathrm{d}t\right]\mathrm{d}\lambda && \text{(29b)}\\
= &\quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\left\{\cos\left[\lambda(t-x)\right]-i\sin\left[\lambda(t-x)\right]\right\}\ \mathrm{d}t\right]\mathrm{d}\lambda\ . && \text{(29c)}
\end{aligned}$$

Now, using the Euler's formula

$$\cos\theta - i\sin\theta = e^{-i\theta} \tag{30}$$

we have:

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\ e^{-i\lambda(t-x)}\ \mathrm{d}t\right]\mathrm{d}\lambda && \text{(31a)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\ e^{-i\lambda t}\ e^{i\lambda x}\ \mathrm{d}t\right]\mathrm{d}\lambda && \text{(31b)}\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\ e^{-i\lambda t}\ \mathrm{d}t\right]\ e^{i\lambda x}\ \mathrm{d}\lambda && \text{(31c)}
\end{aligned}$$

where we have taken the part of the exponential non depending on t out of the inner integral. Finally, we can see the expression in square brackets as a coefficient in the outer integral, and define it as:

$$\boxed{F[f](\lambda)\equiv\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\ e^{-i\lambda t}\ \mathrm{d}t} \tag{32}$$

so to write:

$$\boxed{f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F[f](\lambda)\ e^{i\lambda x}\ \mathrm{d}\lambda} \tag{33}$$

2.3 Comments

In passing from the case of periodic functons, to the case of generic, non-periodic functions, we have used a non-rigorous approach, imagining to start with a period $2l$, and then consider the limit $l\to\infty$. For a periodic function, the Fourier expansion is a series, over infinite terms. For each term, the frequency of the oscillating function (element of the Fourier basis, also known as a *mode*) changes of a fixed amount, $\frac{\pi}{l}$. This frequency increment is inversely proportional to the (half) period l. It means that going from periodic to non-periodic functions, in the frequencies domain we go from discrete to continuous changes. We remark that in both cases the frequencies involved are infinite, i.e. the sum over the frequencies has infinite terms. In figure 1 a drawing shows the link between the frequencies domain and the "direct" domain.

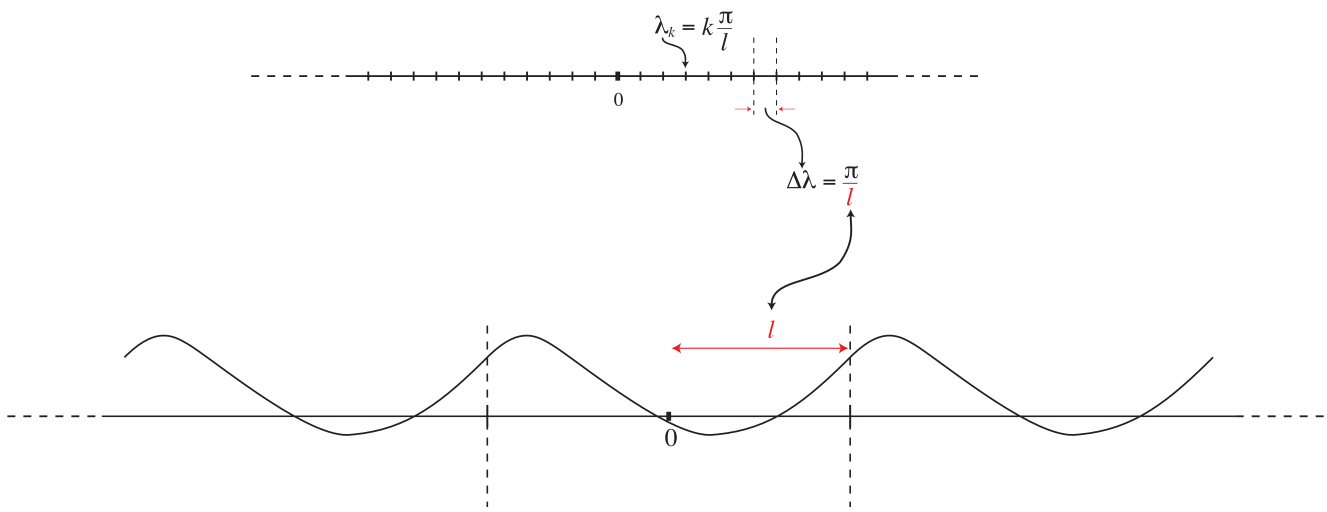

Figure 1: l is the (half) period

References

[1] Andrey Nikolaevich Kolmogorov and Sergey Vladimirovich Fomin. *Elementi di teoria delle funzioni e di analisi funzionale.* Mir, 1980.